

## **TIME SERIES RECONCILIATION THROUGH FLEXIBLE LEAST SQUARES ESTIMATION**

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### **ABSTRACT**

In the paper we optimize a procedure developed recently to reconcile time series by flexible least squares. In the previous version of the procedure the author assumed prior knowledge of the weighting parameter  $\mu$  of Kalaba and Tesfatsion's so-called "incompatibility cost function". In the new version, this parameter is estimated from the sample using an iterative method based on the Newton-Raphson algorithm. We test the improved method by reconciling the monthly growth rates of the Monthly Economic Activity Estimator (EMAE) of Argentina with the quarterly growth rates of the Gross Domestic Product. The results suggest that optimal levels of  $\mu$  would be close to 1 (the value suggested by Kalaba and Tesfatsion as prior value) but estimating  $\mu$  from the sample instead of keeping it fixed at 1 hardly modifies the reconciled series.

**Keywords:** Time series reconciliation, Flexible Least Squares, EMAE, Gross Domestic Product.

**JEL Code:** C130

## CONCILIACIÓN DE SERIES DE TIEMPO A TRAVÉS DE MÍNIMOS CUADRADOS FLEXIBLES

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### RESUMEN

En el artículo optimizamos un procedimiento desarrollado recientemente para conciliar series de tiempo por mínimos cuadrados flexibles. En la versión anterior del procedimiento el autor asumía conocimiento previo del parámetro de ponderación  $\mu$  de la llamada “función de costo de incompatibilidad” de Kalaba y Tesfatsion. En la nueva versión, este parámetro se estima a partir de la muestra mediante un método iterativo basado en el algoritmo de Newton-Raphson. Probamos el nuevo método conciliando las tasas de crecimiento mensual del Estimador de Actividad Económica Mensual (EMAE) de Argentina con las tasas de crecimiento trimestrales del Producto Interno Bruto. Los resultados sugieren que los niveles óptimos de  $\mu$  serían cercanos a 1 (el valor sugerido por Kalaba y Tesfatsion como valor *a priori*) y que estimar  $\mu$  a partir de la muestra en vez de mantenerlo fijo en 1 apenas modifica la serie reconciliada.

**Palabras clave:** conciliación de series de tiempo, mínimos cuadrados flexibles, EMAE, Producto Interno Bruto.

**Código JEL:** C130

## 1 INTRODUCTION

In a recent paper Frank (2017) proposed a linear estimator (based on the flexible least squares criterion) to reconcile time series of different frequencies, understanding by “reconciliation” a rescaling of the high-frequency series to match the low-frequency series as close as possible. Recalling the bibliography: Dagum and Cholette (2006), reconciliation is understood as a procedure that leads to a perfect fit of a high-frequency series to a low-frequency series, assuming that the latter is observed without error while the former is just a rough proxy of the low-frequency series. Frank (2017) discussed the traditional view mainly for four reasons. First, because traditional reconciliation methods do not allow adjustment of high-frequency series in real time but only up to the last available figure of the low-frequency series. Second, because the common practice of government statistical offices in time series reconciliation, as described in INDEC (2016), hides to the public the corrections and updates done in the high-frequency series after the figures of the low-frequency series become available. Third, traditional reconciliation transfers the low-frequency errors to the high-frequency series instead of removing them.<sup>1</sup> Fourth, reconciliation as performed in practice is not informative about the true relationship that links series of different frequencies. The reconciliation procedure proposed by Frank (2017) overcomes these drawbacks although it still requires that the practitioner knows some parameters in advance. In particular, the practitioner should know the parameter  $\mu$  that weighs the error sum of squares of high-frequency series and the sum of squared deviations of the parameters. The procedure also requires that the time period in which the parameters of the model remain constant to be fixed in advance.

## 2 OBJECTIVES

The aim of this paper is to improve the reconciliation procedure proposed in 2017 regarding the knowledge of the aforementioned weighting constant  $\mu$ . The improved estimator will be used to reconcile Argentina’s Monthly Economic Activity Estimator (EMAE, for its acronym in Spanish) with the quarterly Gross Domestic Product (GDP) series and the results will be compared with those obtained in 2017.

## 3 THEORETICAL BACKGROUND

In reference to Kalaba and Tesfatsion’s flexible least squares (FLS) estimator Kalaba and Tesfatsion (1989) and Frank’s estimator Frank (2017) for time series reconciliation. For ease of reading, Kalaba and Tesfatsion’s original formulas in matrix notation should be written. Throughout the paper the notation used is: matrices are written in bold capitals,

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<sup>1</sup>By “traditional reconciliation” we refer to methods widely spread in official statistics offices, such as Chow and Lin (1971), Denton (1971) and Fernández (1981).

vectors in bold lower case, and scalars in italic; unless otherwise mentioned, all vectors are column vectors; greek letters are used for parameters; a tilde on a matrix means that the matrix has been reshaped in a way useful for FLS, as shown next.

### 3.1 Flexible Least Squares Estimation

Consider a vector of observations  $\mathbf{y}$  and a set of explanatory variables  $\mathbf{X}$  such that each  $y_i$  is related linearly to the corresponding row vector  $\mathbf{x}'_i$ . If the parameters of of such relationship vary along observations, the relationship may be written as

$$\mathbf{y} = \tilde{\mathbf{X}} \text{vec}(\mathbf{B}) + \epsilon, \quad \epsilon \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_n),$$

where  $\mathbf{y}$  is the usual  $n \times 1$  vector of observations;  $\tilde{\mathbf{X}}$  is a  $n \times nk$  block diagonal matrix arranged as shown below;  $\text{vec}(\mathbf{B})$  stands for the  $nk \times 1$  vectorized matrix of parameters and  $\epsilon$  is the usual error term of normal i.i.d random variables. Hereinafter we will use the tilde to refer to block diagonal matrices in which each block is a row of the matrix mentioned below the tilde. So,

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{x}'_1 & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \mathbf{x}'_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{x}'_n \end{bmatrix} \quad \text{and} \quad \text{vec}(\mathbf{B}) = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_n \end{bmatrix}.$$

To estimate  $\mathbf{B}$  under the least squares criterion the function to be minimized, which Kalaba and Tesfatsion Kalaba and Tesfatsion (1989) called “incompatibility cost function”, is

$$\begin{aligned} C(\mathbf{B}, \mu, n) &= [\mathbf{y} - \tilde{\mathbf{X}} \text{vec}(\mathbf{B})]' [\mathbf{y} - \tilde{\mathbf{X}} \text{vec}(\mathbf{B})] + \mu \text{vec}(\mathbf{B})' \mathbf{D}' \mathbf{D} \text{vec}(\mathbf{B}) \\ &= \mathbf{y}' \mathbf{y} - 2 \mathbf{y}' \tilde{\mathbf{X}} \text{vec}(\mathbf{B}) + \text{vec}(\mathbf{B})' (\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mu \mathbf{D}' \mathbf{D}) \text{vec}(\mathbf{B}) \end{aligned} \quad (1)$$

where  $\mu$  is a weighting constant and  $\mathbf{D}$  is a  $(n-1)k \times nk$  differentiation matrix so that  $\mathbf{D}' \mathbf{D}$  has the form

$$\mathbf{D}' \mathbf{D} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ -\mathbf{I} & 2\mathbf{I} & -\mathbf{I} & \ddots & & \vdots \\ \mathbf{0} & -\mathbf{I} & 2\mathbf{I} & -\mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & & \ddots & -\mathbf{I} & 2\mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \dots & \dots & \mathbf{0} & -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

and  $\mathbf{I}$  is a  $k \times k$  identity matrix. Although the incompatibility function was originally defined by Kalaba and Tesfatsion as a function of  $\mathbf{B}$ ,  $\mu$  and  $n$ , in practice it is a function of  $\mathbf{B}$  only and conditional on  $\mu$  and  $n$  since the parameters and the sample size are supposed to be given.<sup>2</sup> Then, by solving the first order conditions for the incompatibility cost function we get the so-called “normal” equations and the FLS solution for  $\text{vec}(\mathbf{B})$  which will be unique if and only if  $(\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mu\mathbf{D}'\mathbf{D})$  is a full rank matrix. That is,

$$\text{vec}(\hat{\mathbf{B}})_{\text{OLS}} = (\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mu\mathbf{D}'\mathbf{D})^{-1} \tilde{\mathbf{X}}'\mathbf{y}. \quad (2)$$

### 3.2 Fixing the FLS estimator to estimate $\mu$

Returning to the incompatibility cost function (1) and the solution to the first order condition which leads to the FLS estimator (2). This estimator is conditional on the parameter  $\mu$  which up to now was supposed to be known. However, it is possible to estimate  $\mu$  adding a second condition  $\partial C(\mathbf{B}, \mu)/\partial \hat{\mu} = 0$  in the following fashion.

$$C(\mathbf{B}, \mu) = \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\tilde{\mathbf{X}}\text{vec}(\mathbf{B}) + \text{vec}(\mathbf{B})' (\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mu\mathbf{D}'\mathbf{D}) \text{vec}(\mathbf{B}),$$

Then, the first order condition for  $\mu$  is

$$\frac{\partial C(\mathbf{B}, \mu)}{\partial \hat{\mu}} = \text{vec}(\hat{\mathbf{B}})' \frac{\partial \mathbf{A}(\mu)}{\partial \hat{\mu}} \text{vec}(\hat{\mathbf{B}}) = 0,$$

where  $\mathbf{A}(\mu) = \tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mu\mathbf{D}'\mathbf{D}$ .<sup>3</sup> Replacing  $\text{vec}(\hat{\mathbf{B}})$  by the solution given in (2) a more explicit expression is obtained

$$\frac{\partial C(\mathbf{B}, \mu)}{\partial \hat{\mu}} = \mathbf{y}'\tilde{\mathbf{X}} (\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \hat{\mu}\mathbf{D}'\mathbf{D})^{-1} \mathbf{D}'\mathbf{D} (\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \hat{\mu}\mathbf{D}'\mathbf{D})^{-1} \tilde{\mathbf{X}}'\mathbf{y} = 0. \quad (3)$$

Simple inspection of (3) suggests that (a)  $\mu$  cannot be easily cleared because it is not related linearly with  $\mathbf{X}$ ,  $\mathbf{D}$  and  $\mathbf{y}$ ; and (b) the first order condition does not have a unique solution for  $\mu$ . This last point is shown in the following way:

$$\mathbf{u} = \mathbf{D} (\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \hat{\mu}\mathbf{D}'\mathbf{D})^{-1} \tilde{\mathbf{X}}'\mathbf{y},$$

so that the first-order condition can be written as  $\partial C/\partial \hat{\mu} = \mathbf{u}'\mathbf{u} = 0$ . Without loss of generality, by replacing the matrix  $\mathbf{D}$ , which pre-multiplies the right hand side of the equality, by the square matrix  $\mathbf{D}^* = [\mathbf{D}', \mathbf{0}']'$ . Note that  $\mathbf{D}^*\mathbf{D}^* = \mathbf{D}'\mathbf{D}$  so that the condition  $\mathbf{u}'\mathbf{u} = 0$  is still fulfilled. Then, excluding the trivial solution for  $\tilde{\mathbf{X}}'\mathbf{y} = \mathbf{0}$ , the condition is

<sup>2</sup>The developers of the FLS estimator themselves consider  $\mu$  fixed, even when they reformule this estimator in the context of the Kalman filter Kalaba and Tesfatsion (1990).

<sup>3</sup>Although the matrix between parenthesis was called  $\mathbf{A}$  the reader should be aware that this matrix has nothing to do with other matrices called so elsewhere in the paper.

satisfied if  $\mathbf{u} = \mathbf{0}$  and

$$|\mathbf{D}^*| \left| \left( \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \hat{\mu} \mathbf{D}' \mathbf{D} \right)^{-1} \right| = \frac{|\mathbf{D}^*|}{|\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \hat{\mu} \mathbf{D}' \mathbf{D}|} = 0.$$

But as  $|\mathbf{D}^*|$  is null, it is clear that  $|\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \hat{\mu} \mathbf{D}' \mathbf{D}|$  need not be bounded. In such circumstance, there would not be an optimal  $\hat{\mu}$  to find out. To overcome this drawback we propose a slight amendment to the original incompatibility cost function of Kalaba and Tesfatsion. The proposed amendment is a true weighted average of the error sum of squares and the squared differences among parameters. That is,

$$C(\mathbf{B}, \mu) = (1 - \mu) \mathbf{y}' \mathbf{y} - 2(1 - \mu) \mathbf{y}' \tilde{\mathbf{X}} \text{vec}(\mathbf{B}) + \text{vec}(\mathbf{B})' \left[ (1 - \mu) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mu \mathbf{D}' \mathbf{D} \right] \text{vec}(\mathbf{B}).$$

Then, optimizing  $C$  with respect to  $\text{vec}(\mathbf{B})$  and  $\mu$  results in the first order conditions

$$\frac{\partial C(\mathbf{B}, \mu)}{\partial \text{vec}(\hat{\mathbf{B}})} = -2(1 - \mu) \tilde{\mathbf{X}}' \mathbf{y} + 2 \left[ (1 - \mu) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mu \mathbf{D}' \mathbf{D} \right] \text{vec}(\hat{\mathbf{B}}) = \mathbf{0}$$

so that,

$$\text{vec}(\hat{\mathbf{B}}) = \left[ (1 - \hat{\mu}) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \hat{\mu} \mathbf{D}' \mathbf{D} \right]^{-1} \tilde{\mathbf{X}}' \mathbf{y} (1 - \hat{\mu}) = \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} (1 - \hat{\mu}), \quad (4)$$

and

$$\frac{\partial C(\mathbf{B}, \mu)}{\partial \hat{\mu}} = 2 \mathbf{y}' \tilde{\mathbf{X}} \text{vec}(\hat{\mathbf{B}}) + \text{vec}(\hat{\mathbf{B}})' \left( -\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mathbf{D}' \mathbf{D} \right) \text{vec}(\hat{\mathbf{B}}) - \mathbf{y}' \mathbf{y} = 0.$$

However, the second condition can be manipulated as shown below to get a more friendly expression in order to compute the second derivative, whose usefulness will become apparent shortly.

$$\begin{aligned} \frac{\partial C(\mathbf{B}, \mu)}{\partial \mu} &= 2(1 - \mu) \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} + (1 - \mu)^2 \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left( -\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mathbf{D}' \mathbf{D} \right) \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} - \mathbf{y}' \mathbf{y} \\ &= (1 - \mu) \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left[ 2\mathbf{A} + (1 - \mu) \left( -\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mathbf{D}' \mathbf{D} \right) \right] \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} - \mathbf{y}' \mathbf{y}. \end{aligned}$$

Note that the matrix between brackets may be rewritten as

$$\begin{aligned} 2\mathbf{A} + (1 - \mu) \left( -\tilde{\mathbf{X}}' \tilde{\mathbf{X}} + \mathbf{D}' \mathbf{D} \right) &= 2(1 - \mu) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + 2\mu \mathbf{D}' \mathbf{D} - (1 - \mu) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + (1 - \mu) \mathbf{D}' \mathbf{D} \\ &= (1 - \mu) \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + (1 + \mu) \mathbf{D}' \mathbf{D}. \end{aligned}$$

Therefore,

$$\frac{\partial C(\mathbf{B}, \mu)}{\partial \hat{\mu}} = \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left[ (1 - \hat{\mu})^2 \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + (1 - \hat{\mu}^2) \mathbf{D}' \mathbf{D} \right] \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} - \mathbf{y}' \mathbf{y} = 0.$$

This expression is, however, non-linear with respect to  $\mu$  (recall that  $\mathbf{A}$  is also a function of  $\mu$ ), so ought to solve it by a computational method such as the Newton-Raphson's iterative procedure (Greene, 2008, p. 1070). The reader may find a brief explanation of this method in the appendix at the end of the paper. Then, in the context of the incompatibility cost function  $C^*(\mathbf{B}, \mu)$ , the recursion relationship of Newton-Raphson's method may be restated as

$$\hat{\mu}_{(h+1)} = \hat{\mu}_{(h)} - \left[ \frac{\partial^2 C^*(\mathbf{B}, \mu)}{\partial \hat{\mu}_{(h)}^2} \right]^{-1} \frac{\partial C^*(\mathbf{B}, \mu)}{\partial \hat{\mu}_{(h)}}, \quad (5)$$

where the superscript between parenthesis refers to the iteration number. To compute  $\hat{\mu}$  in this fashion we must compute the second derivative of  $C^*(\mathbf{B}, \mu)$  with respect to  $\mu$ . To that end, we call  $\mathbf{C} = (1 - \mu)^2 \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + (1 - \mu^2) \mathbf{D}' \mathbf{D}$ . Then,

$$\frac{\partial^2 C(\mathbf{B}, \mu)}{\partial \mu^2} = \mathbf{y}' \tilde{\mathbf{X}} \left[ \frac{\partial \mathbf{A}^{-1}}{\partial \mu} \mathbf{C} \mathbf{A}^{-1} + \mathbf{A}^{-1} \left( \frac{\partial \mathbf{C}}{\partial \mu} \mathbf{A}^{-1} + \mathbf{C} \frac{\partial \mathbf{A}^{-1}}{\partial \mu} \right) \right] \tilde{\mathbf{X}}' \mathbf{y}.$$

The reader may check that  $\partial \mathbf{C} / \partial \mu = -2\mathbf{A}$  and  $\mathbf{C} = (1 - \mu)(\mathbf{A} + \mathbf{D}' \mathbf{D})$ . In the appendix we develop the derivative of  $\mathbf{A}^{-1}$  with respect to  $\mu$ . Replacing the derivatives involving  $\mathbf{A}^{-1}$  and  $\mathbf{C}$  and arranging the terms conveniently we get

$$\begin{aligned} \frac{\partial^2 C(\mathbf{B}, \mu)}{\partial \mu^2} &= \mathbf{y}' \tilde{\mathbf{X}} \left[ \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1} \mathbf{C} \mathbf{A}^{-1} - \mathbf{A}^{-1} \left( 2\mathbf{I}_{nk} + \mathbf{C} \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1} \right) \right] \tilde{\mathbf{X}}' \mathbf{y} \\ &= \mathbf{y}' \tilde{\mathbf{X}} \left( \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1} \mathbf{C} \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{C} \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1} - 2\mathbf{A}^{-1} \right) \tilde{\mathbf{X}}' \mathbf{y}. \end{aligned}$$

Note that every term between brackets is a symmetric matrix, so the expression above can be reduced to

$$\frac{\partial^2 C(\mathbf{B}, \mu)}{\partial \mu^2} = 2 \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left( \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1} \mathbf{C} - \mathbf{A} \right) \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y}. \quad (6)$$

Writing  $\partial \mathbf{A} / \partial \mu$  in terms of  $\tilde{\mathbf{X}}$  and  $\mathbf{D}$ , and  $\mathbf{C}$  in terms of  $\mathbf{A}$  and  $\mathbf{D}$ , (6) is

$$\frac{\partial^2 C(\mathbf{B}, \mu)}{\partial \mu^2} = 2 \mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left[ (1 - \mu) (\mathbf{D}' \mathbf{D} - \tilde{\mathbf{X}}' \tilde{\mathbf{X}}) (\mathbf{I}_{nk} + \mathbf{A}^{-1} \mathbf{D}' \mathbf{D}) - \mathbf{A} \right] \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y}.$$

Thus, the first order condition is now able to be solved for  $\hat{\mu}$  by Newton-Raphson's recursion. By replacing the first and second derivatives in (5) results in

$$\hat{\mu}_{(h+1)} = \hat{\mu}_{(h)} - \frac{1}{2} \frac{\mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left[ (1 - \hat{\mu}_{(h)})^2 \tilde{\mathbf{X}}' \tilde{\mathbf{X}} + (1 - \hat{\mu}_{(h)}^2) \mathbf{D}' \mathbf{D} \right] \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y} - \mathbf{y}' \mathbf{y}}{\mathbf{y}' \tilde{\mathbf{X}} \mathbf{A}^{-1} \left[ (1 - \hat{\mu}_{(h)}) (\mathbf{D}' \mathbf{D} - \tilde{\mathbf{X}}' \tilde{\mathbf{X}}) (\mathbf{I}_{nk} + \mathbf{A}^{-1} \mathbf{D}' \mathbf{D}) - \mathbf{A} \right] \mathbf{A}^{-1} \tilde{\mathbf{X}}' \mathbf{y}}.$$

Although this expression looks quite cumbersome, it may be rewritten in a more friendly fashion just calling  $\mathbf{G}_1 = \tilde{\mathbf{X}}'\tilde{\mathbf{X}}$ ,  $\mathbf{G}_2 = \mathbf{D}'\mathbf{D}$ ,  $\mathbf{A} = (1 - \mu)\mathbf{G}_1 + \mu\mathbf{G}_2$  and  $\mathbf{d} = \mathbf{A}^{-1}\tilde{\mathbf{X}}'\mathbf{y}$ . Then,

$$\hat{\mu}_{(h+1)} = \hat{\mu}_{(h)} - \frac{1}{2} \frac{\mathbf{d}'\mathbf{C}\mathbf{d} - \mathbf{y}'\mathbf{y}}{\mathbf{d}'[(\mathbf{G}_2 - \mathbf{G}_1)\mathbf{A}^{-1}\mathbf{C} - \mathbf{A}]\mathbf{d}}. \quad (7)$$

### 3.3 Time Series Reconciliation through FLS

As already mentioned, Frank (2017) proposed a FLS estimator for time series reconciliation. The goal was to fit one or more high-frequency series to a low-frequency series assuming that both sets are observed with error. To that end, the aforementioned author proposed a data generating process expressible by two sets of equations.

$$\begin{aligned} \mathbf{H}\bar{\mathbf{y}} &= \tilde{\mathbf{Z}}\text{vec}(\mathbf{B}) + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\boldsymbol{\Omega}), \quad \mathbf{H} = \mathbf{I}_m \otimes \mathbf{1}_q, \quad \bar{\mathbf{y}} = \mathbf{P}\mathbf{y} \\ \text{vec}(\hat{\mathbf{B}}_0) &= \text{vec}(\mathbf{B}) + \boldsymbol{\nu}, \quad \boldsymbol{\nu} \sim N[\mathbf{0}, \sigma_\nu^2(\mathbf{I}_n \otimes \boldsymbol{\Psi})]. \end{aligned} \quad (8)$$

In the first set, the high-frequency series in  $\mathbf{Z}$  are related linearly to the “expanded” version of the low-frequency series  $\bar{\mathbf{y}}$  by fixed, but time-varying, parameters. The expanded low-frequency series  $\mathbf{H}\bar{\mathbf{y}}$  is just a time series in which each observation appears repeated  $q$  times to match the length of the high-frequency series.  $\mathbf{H}$  is an “expansion matrix” while  $\mathbf{P} = \mathbf{H}'/q$ , is an  $m \times mq$  matrix that averages the elements of the unobservable time series  $\mathbf{y}$ .  $\tilde{\mathbf{Z}}$  is an  $n \times nk$  matrix of high-frequency series with its rows placed blockwise, as in  $\tilde{\mathbf{X}}$ . Perhaps, a clearer way of defining the relationship between  $\tilde{\mathbf{Z}}$  and  $\mathbf{y}$  would be

$$\bar{\mathbf{y}} = \tilde{\mathbf{P}}\tilde{\mathbf{Z}}\text{vec}(\mathbf{B}) + \boldsymbol{\xi}, \quad \boldsymbol{\xi} \sim N(\mathbf{0}, \sigma^2\boldsymbol{\Phi}) \quad \text{and} \quad \boldsymbol{\Phi} = \mathbf{P}\boldsymbol{\Omega}\mathbf{P}'.$$

This relationship, however, is useless for reconciling time series in real time since the number of rows of  $\tilde{\mathbf{Z}}$  must match the length of the low-frequency series, that is,  $n = mq$ . The first set of equations in (8) has a random error term with unknown covariance  $\sigma^2\boldsymbol{\Omega}$ . An explanation on how to estimate  $\sigma^2\boldsymbol{\Omega}$  will be given below. The second set introduces a prior estimate  $\hat{\mathbf{B}}_0$  of the true parameters of the first set. This estimate is supposed to be unbiased with a known block-diagonal covariance structure  $\mathbf{I}_n \otimes \boldsymbol{\Psi}$ . Under this specification, the incompatibility cost function to be optimized is the following extended version of Kalaba and Tesfatsion’s original incompatibility function.

$$\begin{aligned} C^*(\mathbf{B}|\mu) &= [\mathbf{H}\bar{\mathbf{y}} - \tilde{\mathbf{Z}}\text{vec}(\mathbf{B})]'(\sigma^2\boldsymbol{\Omega})^{-1}[\mathbf{H}\bar{\mathbf{y}} - \tilde{\mathbf{Z}}\text{vec}(\mathbf{B})] + \frac{\mu}{\sigma^2}\text{vec}(\mathbf{B})'\mathbf{D}'\mathbf{D}\text{vec}(\mathbf{B}) + \\ &+ \frac{1}{\sigma_\nu^2}[\text{vec}(\tilde{\mathbf{B}}) - \text{vec}(\mathbf{B})]'(\mathbf{I}_n \otimes \boldsymbol{\Psi})^{-1}[\text{vec}(\hat{\mathbf{B}}_0) - \text{vec}(\mathbf{B})]. \end{aligned} \quad (9)$$



The first order condition for minimizing  $C^*$  conditional to  $\mu$ ,  $\sigma^2$ ,  $\sigma_v^2$  and  $\widehat{\mathbf{B}}_0$  is

$$\begin{aligned} \frac{\partial C^*(\mathbf{B}|\mu)}{\partial \text{vec}(\widehat{\mathbf{B}})} &= -2 \frac{1}{\sigma^2} \tilde{\mathbf{Z}}' \Omega^{-1} \mathbf{H} \bar{\mathbf{y}} + 2 \frac{1}{\sigma^2} \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} \text{vec}(\widehat{\mathbf{B}}) + 2 \frac{\mu}{\sigma^2} \mathbf{D}' \mathbf{D} \text{vec}(\widehat{\mathbf{B}}) + \\ &+ 2 \frac{1}{\sigma_v^2} (\mathbf{I}_n \otimes \Psi^{-1}) \left[ \text{vec}(\widehat{\mathbf{B}}) - \text{vec}(\widehat{\mathbf{B}}_0) \right] = \mathbf{0}. \end{aligned}$$

Rearranging terms, calling  $\alpha = \sigma_v^2/\sigma^2$  and manipulating conveniently the expression above the solution for  $\text{vec}(\widehat{\mathbf{B}})$  is

$$\begin{aligned} \text{vec}(\widehat{\mathbf{B}}) &= \left[ \mathbf{I}_{nk} + \frac{1}{\alpha} \left( \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D} \right)^{-1} (\mathbf{I}_n \otimes \Psi^{-1}) \right]^{-1} \text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}} + \\ &+ \left[ \mathbf{I}_{nk} + \alpha (\mathbf{I}_n \otimes \Psi^{-1})^{-1} \left( \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D} \right) \right]^{-1} \text{vec}(\widehat{\mathbf{B}}_0). \end{aligned} \quad (10)$$

where  $\text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}}$  is the generalized version of the FLS estimator given in (2).

$$\text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}} = \left( \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D} \right)^{-1} \tilde{\mathbf{Z}}' \Omega^{-1} \mathbf{H} \bar{\mathbf{y}}.$$

Notwithstanding, if  $\text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}}$  were a true GLS estimator, the term containing  $\mu$  should appear multiplied by  $\alpha$ . However, the formula given above is correct because actually matrix  $\tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D}$  was introduced to reformulate  $\text{vec}(\widehat{\mathbf{B}})$  as a function of  $\text{vec}(\widehat{\mathbf{B}})_{\text{GLS}}$  and  $\text{vec}(\widehat{\mathbf{B}}_0)$ .

The reader may check that the weighting matrices that pre-multiply  $\text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}}$  and  $\text{vec}(\widehat{\mathbf{B}}_0)$  add up to  $\mathbf{I}_{nk}$  so that the current estimate might be interpreted as a weighted average of the estimate of  $\mathbf{B}$  known before sampling and that computed from the sample. Frank (2017) provided an alternative expression to compute the reconciled series  $\widehat{\mathbf{H}} \bar{\mathbf{y}}$  when all the time periods share the same single prior  $\hat{\mathbf{b}}_0$ . The expression is as follows

$$\begin{aligned} \widehat{\mathbf{H}} \bar{\mathbf{y}} | \mu, \alpha &= \tilde{\mathbf{Z}}^* \left[ \mathbf{I}_{nk} + \frac{1}{\alpha} \left( \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D} \right)^{-1} (\mathbf{I}_n \otimes \Psi^{-1}) \right]^{-1} \text{vec}(\widehat{\mathbf{B}})_{\text{FGLS}} + \\ &+ \tilde{\mathbf{Z}}^* \left[ \mathbf{I}_{nk} + \alpha (\mathbf{I}_n \otimes \Psi^{-1})^{-1} \left( \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \mu \mathbf{D}' \mathbf{D} \right) \right]^{-1} \hat{\mathbf{b}}_0. \end{aligned} \quad (11)$$

To make this estimation feasible it is necessary to replace  $\Omega$ ,  $\Psi$  and  $\hat{\mathbf{b}}_0$  by proper *proxies*. In fact, Frank's paper focused on finding a good proxy for  $\Omega$ . The one found is a Toeplitz matrix  $\mathbf{W}$  whose first column  $\mathbf{w}$  is the linear programming solution satisfying

$$\min_{\mathbf{w}} \{ \mathbf{1}' \mathbf{w} \} \quad \text{subject to} \quad w_1 = 1, \mathbf{A}_1 \mathbf{w} = q^2 \left[ \frac{\phi_1}{2}, \phi'_{i>1} \right]', \text{ and } \mathbf{A}_2 \mathbf{w} \geq \mathbf{0}_{2n-1},$$

where  $\mathbf{A}_1$  is a set of  $m$  linear constraints (see appendix in Frank (2017)) that relates the elements of  $\Omega$  to those of a covariance matrix  $\Phi$  and  $\mathbf{A}_2$  is a set of  $2n - 1$  constraints introduced to guarantee that each  $w_i \geq 0$  and the difference  $w_i - w_{i+1} \geq 0$ , as are supposed

to be the typical covariance structures with positive autocorrelation. The covariance structure that arises from  $\mathbf{w}$  is completely non-parametric and can be plugged in directly in (10) if  $\Omega$  were non-singular. Anyway, to avoid numerical unstabilities if  $\mathbf{W}$  were ill-conditioned, Frank assumed an AR(1) covariance structure for  $\Omega$  and used  $\mathbf{w}$  to compute the single parameter of this structure,  $\rho$ .

The previous result is extensible to the FGLS estimator by just calling  $\tilde{\mathbf{X}} = \Omega^{-\frac{1}{2}}\tilde{\mathbf{Z}}$  and  $\mathbf{y} = \Omega^{-\frac{1}{2}}\mathbf{H}\tilde{\mathbf{y}}$ . However, it should be pointed out that in the new incompatibility cost function the parameter  $\mu$  has a different meaning than the one in the original function. In fact, the relationship between  $\mu$  in Kalaba and Tesfatsion's estimator and  $\mu^*$  in (4) is exactly  $\mu = \mu^*/(1 - \mu^*)$ , provided  $\mu^* \neq 1$ . So two courses of action may be followed to compute the reconciled series  $\widehat{\mathbf{H}\tilde{\mathbf{y}}}$ . One course would be to compute directly

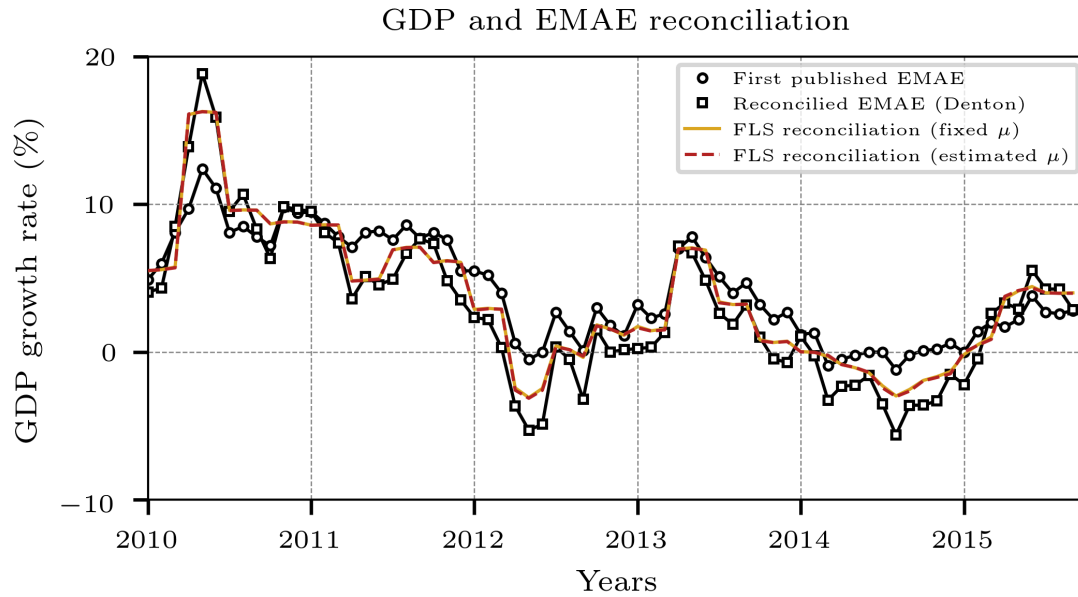
$$\begin{aligned} \widehat{\mathbf{H}\tilde{\mathbf{y}}}|\Omega, \alpha = \tilde{\mathbf{Z}}^* \left\{ \mathbf{I}_{nk} + \frac{1}{\alpha} \left[ (1 - \hat{\mu}^*) \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \hat{\mu}^* \mathbf{D}' \mathbf{D} \right]^{-1} (\mathbf{I}_n \otimes \Psi^{-1}) \right\}^{-1} \text{vec}(\widehat{\mathbf{B}}) + \\ + \mathbf{Z}^* \left\{ \mathbf{I}_{nk} + \alpha (\mathbf{I}_n \otimes \Psi^{-1})^{-1} \left[ (1 - \hat{\mu}^*) \tilde{\mathbf{Z}}' \Omega^{-1} \tilde{\mathbf{Z}} + \hat{\mu}^* \mathbf{D}' \mathbf{D} \right] \right\}^{-1} \hat{\mathbf{b}}_0, \end{aligned} \quad (12)$$

where  $\hat{\mu}^*$  is the estimation that arises from Newton-Raphson's iterative procedure, performed when computed  $\text{vec}(\widehat{\mathbf{B}})$ . The other course of action would be to compute  $\mu$  in a preliminar round, but assuming  $\Omega = \mathbf{I}$  and  $\alpha = 1$ , and then proceed in the usual way as if  $\mu$  is known.

#### 4 EXAMPLE: RECONCILING EMAE WITH THE QUARTERLY GDP

Next, the reconciliation of EMAE's growth rates with the quarterly GDP growth rates for the period ranging from January 2010 to September 2015 is presented. The period January 2007 to December 2009 was set aside to compute the *prior* estimates  $\hat{\mathbf{b}}_0$ . Note that the growth rates to be reconciled are EMAE's first published growth rates without any further revision, so the EMAE series used in the example that follows is not the downloadable version available at INDEC's official homepage, but one compiled by the author based on INDEC's press releases. Both the monthly and the quarterly series chosen to exemplify the proposed reconciliation procedure are the same that those used in Frank (2017) in order to let the results be fully comparative.

The figure below shows EMAE's first reported interannual growth rates, the quarterly GDP interannual growth rates reconciled with EMAE's last revised series (following Denton's method) and overlapped the reconciled series computed according to the proposal explained in Frank (2017) and the improved version explained above. Simple inspection of the graph shows that the FLS series fits better the quarterly series than the GLS series, although both alternatives perform accurately. The graph also shows that the



**Figure 1.** EMAE's first reported growth rates, quarterly GDP growth rates reconciled with EMAE's last revised series by Denton's method and FLS reconciliation.

proposed procedure returns a softer monthly series and avoids spurious values at the end of the series that form a typical outcome of traditional reconciliation methods. Recall that the common practice to overcome this problem is to forecast the low-frequency series one period ahead and then reconcile the whole series as if all the figures were obtained by the same data generating process. This practice, however, also requires forecasts of monthly future values to match the period covered by the quarterly series. So the accuracy of the reconciliation procedure, at least at the end of the series, relies heavily on the method chosen to forecast future periods. This issue might obscure the whole reconciliation method.

Although the graphical inspection does not reveal conspicuous differences between the series reconciled with fixed and estimated  $\mu$ , a statistical test was conducted to check the convenience of estimating  $\mu$ . To that end, the difference between the reconciled series was modeled as a SARIMA  $(p,d,q)(P,D,Q)$  process with the program X-13 ARIMA-SEATS.<sup>4</sup> The program was used to adjust 576 possible specifications, corresponding to all possible combinations of  $p, q = \{0, 1, 2, 3\}$ ,  $P, Q = \{0, 1, 2\}$  and  $d, D = \{0, 1\}$ , and select the one with the lowest MAPE (mean absolute percentage error), as long as this does not exceed 15% and does not show evidence of overdifferentiation in the seasonal and non-seasonal component. If none of the models fits the data, it may be concluded that the data follow a white noise process and there is not a significant difference between the series.

<sup>4</sup>X-13 ARIMA-SEATS is a free software developed by the U.S. Census Bureau downloadable from <https://www.census.gov/data/software/x13as.html>.

## 5 CONCLUDING REMARKS

The convenience of reconciling low and high frequency series by FLS instead of the traditional Denton method was discussed in Frank (2017) and it is mainly supported by the fact that the estimation error cannot always be attributed to lack of information in the construction of the high frequency series. In this paper the focus is put on the estimation of the parameter  $\mu$  (assuming that both series are noisy), a topic that seems to have been neglected in the literature referring to the FLS estimator. During the course of the investigation it became evident that the so-called “incompatibility cost function” proposed by Kalaba and Tesfatsion cannot be optimized with respect to  $\mu$  because the parametric space of  $\mu$  is unbounded. Instead, a slightly modified version of the incompatibility cost function was proposed to bound  $\mu$  in the interval  $\mu \in (0, 1)$ . The possibility that  $\mu = 0$  or  $\mu = 1$  was explicitly excluded because these values lead to singular matrices  $\mathbf{A}$ , or degeneracy in the estimation of the type

$$\lim_{\mu \rightarrow 1} \text{vec}(\mathbf{B}) = \lim_{\mu \rightarrow 1} [(1 - \mu)\mathbf{X}'\mathbf{X} + \mu\mathbf{D}'\mathbf{D}]^{-1} \mathbf{X}'\mathbf{y}(1 - \mu) = \mathbf{0}$$

The new reconciled series of quarterly and monthly growth rates did not show a major improvement compared with a previous series calculated keeping  $\mu$  fixed in 1, as the difference between both series could not be fitted with any of the tested SARIMA models. However, it is yet premature to state that estimating  $\mu$  from a sample does not improve substantially the reconciliation of time series assuming that  $\mu$  is fixed because the value of  $\mu$  in our particular example was 0.4626, equivalent to 0.8608 in the traditional scale, which is quite close to  $\mu = 1$  often used as a default value in flexible estimation.

Besides, it was found that starting the Newton-Raphson algorithm with values close to 0 or 1, convergence cannot always be guaranteed. These failures are attributable, mostly, to the slowness of the algorithm and, in a lesser extent, to numerical instabilities that hindered the inversion of matrices  $\mathbf{A}$ . These latter cases could have been avoided by replacing problematic inverses of  $\mathbf{A}$  with approximate inverses obtained by SVD-inversion. However, in the exercise all matrices were inverted by pure Gaussian elimination and back-substitution in order to detect this type of numerical drawbacks. When the initial values chosen were not close to 0 or 1, and  $\delta < 0.001$ , the convergence was reached in a few steps.

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## ANEXOS

### A The Newton-Raphson Method

A brief explanation of Newton-Raphson's recursive procedure for finding the roots of a real-valued function is as follows. Recall first the Taylor decomposition of a function  $f(\mathbf{x})$  in the neighborhood of a certain point  $\mathbf{x}_m$ .

$$f(\mathbf{x}) = f(\mathbf{x}_m) + (\mathbf{x} - \mathbf{x}_m)' \frac{\partial f(\mathbf{x}_m)}{\partial \mathbf{x}_m} + \frac{1}{2} (\mathbf{x} - \mathbf{x}_m)' \frac{\partial^2 f(\mathbf{x}_m)}{\partial \mathbf{x}_m \mathbf{x}_m'} (\mathbf{x} - \mathbf{x}_m) + \dots$$

In particular, at some point  $\mathbf{x}_{m+1}$ , the function  $f(\mathbf{x})$  can be approximated by the first two terms of the Taylor series

$$f(\mathbf{x}_{m+1}) \approx f(\mathbf{x}_m) + (\mathbf{x}_{m+1} - \mathbf{x}_m)' \frac{\partial f(\mathbf{x}_m)}{\partial \mathbf{x}_m}$$

and, in case  $f(\mathbf{x}_{m+1}) = 0$

$$\mathbf{x}_{m+1} \approx \mathbf{x}_m - \left[ \frac{\partial f(\mathbf{x}_m)}{\partial \mathbf{x}_m} \right]^{-1} f(\mathbf{x}_m). \quad (13)$$

If  $f(\mathbf{x})$  were a function to be optimized, for instance the errors sum of squares of a linear model, expression (13) is useful to find recursively the estimated parameters of the model. In terms of a standard regression (13) may be rewritten as

$$\mathbf{b}_{m+1} = \mathbf{b}_m - \left[ \frac{\partial^2 L}{\partial \mathbf{b}_m \mathbf{b}_m'} \right]^{-1} \frac{\partial L}{\partial \mathbf{b}_m}.$$

where  $\partial L / \partial \mathbf{b}_m$  is the first derivative of the error sum of squares or the log likelihood function, and  $m$  is the iteration number.

### B Some results in matrix calculus

Recall that for any non-singular square matrix  $\mathbf{A}$  there exists a unique matrix  $\mathbf{A}^{-1}$  such that  $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ . Then, if  $\mathbf{A}$  is a function of  $\mu$ , deriving this identity with respect to  $\mu$  on both sides of the equality yields,

$$\frac{\partial \mathbf{A}^{-1}}{\partial \mu} \mathbf{A} + \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} = \mathbf{0}, \quad \text{so that} \quad \frac{\partial \mathbf{A}^{-1}}{\partial \mu} = -\mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \mu} \mathbf{A}^{-1}.$$