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DENSITY ESTIMATION USING BOOTSTRAP QUANTILE VARIANCE AND QUANTILE-MEAN COVARIANCE

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MODELO DE COMPONENTES DE ERRORES PARA REDES: ESPECIFICACIÓN Y CONTRASTES

ABSTRACT

We propose two novel bootstrap density estimators based on the quantile variance and the quantile-mean covariance. We review previous developments on quantile-density estimation and asymptotic results in the literature that can be applied to this case. We conduct Monte Carlo simulations for dierent data generating processes, sample sizes, and parameters. The estimators perform well in comparison to benchmark non-parametric kernel density estimator. Some of the explored smoothing techniques present lower bias and mean integrated squared errors, which indicates that the proposed estimator is a promising strategy.

RESUMEN

Evaluamos dos estimadores de densidades basados en la varianza y la covarianza entre media y varianza estimados por bootstrap. Revisamos otros desarrollos de estimadores de densidad relacionados con cuantiles. Las simulaciones de Monte Carlo para distintos procesos generadores de datos, tamaños de muestra, y otros parámetros muestran que los estimadores tienen buena performance en comparación con el estimador no paramétrico de kernel. Algunas de las técnicas de suavizamiento tienen menor error cuadrático medio integrado y sesgo, lo que indica que los estimadores propuestos son una estrategia promisoria.

Keywords: Density Estimation - Quantile Variance - Quantile-Mean Covariance - Bootstrap
Palabras claves: Estimación de Densidades - Varianza de Cuantiles - Covarianza entre Media y Cuantiles - Bootstrap

JEL Codes: C13, C14, C15 y C46

1 Introduction

A well known result is that if the continuous random variable X has distribution function F, density f and median $q_{0.5}$, then the sample variance of $\hat{q}_{0.5}$, the median estimator, has asymptotic variance $\frac{1}{4f(q_{0.5})^2}$. According to Stigler (1973), this result together with the asymptotic properties of the median estimator was firstly stated by Laplace in 1818. A similar result applies for any quantile $\tau \in (0, 1)$: let q_τ (where $q_\tau := F^{-1}(\tau) = \inf\{x : F(x) \geq \tau\}$) and $\hat{q}_\tau$ be the population and sample τ-quantile, respectively, then, the asymptotic variance of the sample quantile is equal to $\frac{\tau(1-\tau)}{f_X(q_\tau)^2}$ (see, e.g., Moore, 1969). Ferguson (1999) extended these results and proved that the sample quantile and the sample average have an asymptotic covariance equal to $\frac{\varpi(\tau)}{f(q_\tau)}$, where $\varpi(\tau) = E[\rho_\tau(X - q_\tau)]$ and $\rho_\tau(u) = u\{\tau - 1[u \leq 0]\}$ is the quantile check function in Koenker and Bassett (1978). Overall, the above known results indicate that density functions f and moments of the sample quantiles are closely linked to each other.

As stated by Koenker (1994), given that $f(q_\tau)$ reflects the density of observations near the quantile τ, it is not surprising that the asymptotic precision of quantile estimates depend on the reciprocal of a density function evaluated at the quantile of interest. This idea of studying the locations where information is more *sparse* had led Tukey (1965) to name the *sparsity function*, s_τ, as the inverse of the density function, also referred as *quantile* *density function* by Parzen (1979). More precisely, it can be stated that $s_\tau := \frac{dq_\tau}{d\tau} = q'_\tau = \frac{1}{f(q_\tau)}$.

Based on the asymptotic properties of the sample quantile variance bootstrap estimation derived by Babu (1986) and Hall and Martin (1988), and the asymptotic quantile-mean covariance derived by Ferguson (1999), we explore the reverse problem of first estimating the sample quantile variance and sample quantile-mean covariance using bootstrap, and then using them to estimate the density function. This paper explores different smoothing techniques to improve these estimators.

Using Monte Carlo simulations, we show that the estimators perform well in comparison to the benchmark kernel density estimator. Some of the explored smoothing techniques have lower bias and mean integrated squared errors, which indicate that the proposed estimator is a promising strategy.

This paper is organized as follows. Section 2 presents a short literature review of density estimation based on estimated quantiles. Sections 3 and 4 develop the two density estimators proposed in this paper, and Section 5 the smoothing techniques applied to those. Section 6 presents Monte Carlo simulations. Section 7 concludes.

2 Short literature review on recent advances on density estimation using quantiles

The study of ordered statistics, i.e. the i_{th} statistic $X_{(i)}$ coming from a set of realizations of random variables $X_1 \ldots X_n$ ordered such that $X_{(1)} \leq \ldots \leq X_{(n)}$, has a long tradition and a wide range of applications (see David and Nagaraja, 2003, for a wide introductory coverage on the subject). Among the ordered statistics, the use of percentiles or quantiles were pioneered by Galton (1889),[1] who computed medians and quartiles of conditional distributions of the height of sons given the height of their parents. Since then, quantile's properties were further developed with remarkable contributions from Tukey (1965), Pyke (1965), Parzen (1979), and Koenker and Bassett (1978) among others.

The function q_τ, denoted as the *representative function* by Tukey (1965) or as *quantile function* by Parzen (1979), received increased attention over the last decades. Parzen (2004) enumerate many of the benefits of doing statistics in the *"quantile way"* and advocates for the potential of this perspective for achieving an unification of the statistical methods.[2] Whatever the optimism's degree of this proposition, the fact is that due to the growing use of quantile

[1] However, many properties of quantiles were developed before. See for example Hald (1998) for a review on developments since 1750 and Stigler (1973) for Laplace's derivation of the asymptotic distribution of the median.

[2] From Parzen (2004, p.1): *"I teach that statistics (done the quantile way) can be simultaneously frequentist and Bayesian, confidence intervals and credible intervals, parametric and nonparametric, continuous and discrete data."*

functions over the last decades, a growing number of $\hat{q}(\tau)$ estimators and different related applications emerged.

The limiting distributions for the sample quantiles, $\hat{q}_\tau$, was attributed to Mosteller et al. (1946); an intuitive proof is presented in Moore (1969); also refer to conventional textbook proofs in Gilchrist (1980, p.77) and David and Nagaraja (2003, p.287). This implies that by estimating $\hat{f}(q_\tau)$, confidence intervals for $\hat{q}_\tau$ can be constructed, and viceversa. As stated by Koenker (1994), given that $f(q_\tau)$ reflects the density of observations near the quantile of interest, it is not surprising that asymptotic precision of quantile estimates depends on the reciprocal of a density function evaluated at the quantile of interest. This idea of locations where information is more *sparce* had led Tukey (1965) to name *sparsity function* at the inverse of the density function, also referred as *quantile density function* by Parzen (1979).

Parzen (1979) introduced the kernel estimator for $\hat{q}_\tau$, and it was subsequently studied by Falk (1986), Csörgő et al. (1991), and Cheng (1998) among others. This estimator is in the spirit of Siddiqui (1960) where the quantile was estimated using a kernel, but it has the advantage of being estimated using a single step by running a kernel smoother through consecutive order statistics. Further advances on this can be found in Jones (1992), Wang et al. (2012), Soni et al. (2012), Chesneau et al. (2016), Mnatsakanov and Sborshchikovi (2018), and Saadi et al. (2019).

Simultaneously, many efforts have been applied to directly estimate second moments of point quantiles without the need of firstly estimating density function. For instance, Maritz and Jarrett (1978) pointed out the poor small sample performance of the median variance estimations using Laplace's asymptotic result $\frac{1}{4f(q_{0.5})^2}$. As an alternative, they proposed an estimator based on incomplete beta functions capable of estimating variance for any given order statistic or linear combination of them. Also, Efron (1979) inspired on the *jacknife* method (Miller, 1974) proposed the *bootstrap* for estimating not only variance, but also many other moments. Subsequently, Babu (1986) proved consistency and Hall and Martin (1988) derived the rate of convergence of the bootstrap quantile variance estimator.

Furthermore, Hall and Martin (1988) remarked the fact that, since quantile variance estimation is tantamount to density estimation, their findings apply equally well to a density estimator in the way we review in detail in the following Section 3. Additionally, since the introduction of bootstrap to this area, many advances had been done on variance estimation exploring the three approaches proposed by Efron's seminal work.[3]

Finally, Ferguson (1999) derived the asymptotic joint distribution of the sample mean and any sample quantile. We review this result in Section 4.

[3] We will not discuss them, but limit ourselves to indicate that our proposed approach for density estimation is flexible enough to be implemented, and presumably improved, by any overperfoming method for variance and covariance estimation such as those proposed by Sheater (1986), Huang (1991), Janas (1993), Rao et al. (1997), Hutson and Ernst (2000), Ho and Lee (2005), Cheung and Lee (2005), and Alin et al. (2017).

Bera et al. (2014) employed this results to test the equality of mean and quantile effects on quantile regressions. Also, Bera et al. (2016) and Alejo et al. (2016) showed that for the normal distribution the asymptotic covariance between the sample mean and quantile is constant across all quantiles and employed this fact for developing a new normality test. However, to the best of our knowledge there are no results about the convergence rate of quantile-mean covariance estimation using bootstrap, neither an application of this results to density estimations. We expect that based on the promising results from our Monte Carlo simulations, further research on this topic will be done.

3 Quantile variance density estimator

Let $(X_1, ..., X_n)$ be an $i.i.d.$ sample with distribution function F, density f, mean $E(X) = \mu$ and finite variance $Var(X) = \sigma^2$. Let $0 < \tau < 1$ and let $q_\tau = \inf\{x : F(x) > \tau\}$ denote the τ-quantile of X, so that $F(q_\tau) = \tau$. Assume that the density $f(x)$ is continuous and positive at $x = q_\tau$ for all $\tau \in (0, 1)$. Further consider the assumptions of Theorem 2.2 in Hall and Martin (1988).

Let $\hat{q}_\tau$ denote the sample τ-quantile. Then, as $n \to \infty$, (Mosteller et al., 1946; Moore, 1969):

$$\sqrt{n}(\hat{q}_\tau - q_\tau) \xrightarrow{d} N\left(0, \frac{\tau(1-\tau)}{f(q_\tau)^2}\right). \tag{1}$$

The quantile variance function, $Var(q_\tau)$, is the asymptotic variance of the sample quantiles, i.e., $\hat{q}_\tau$,

$$Var(q_\tau) \equiv \lim_{n\to\infty} nVar(\hat{q}_\tau). \tag{2}$$

From eqs. (1) and (2) we have

$$f(q_\tau) = \sqrt{\frac{\tau(1-\tau)}{Var(q_\tau)}}. \tag{3}$$

Then quantile variance estimator of $f(q_\tau)$ can be constructed using a consistent estimator of $V(q_\tau)$. We evaluate the non-parametric bootstrap estimator which takes B random sub-samples of size n out of n observations. This estimator has been studied by Hall and Martin (1988).

Consider the following algorithm:

1. Consider B bootstrap sub-samples of size n, $\{X_i^b\}_{i=1}^n, b = 1, 2, ..., B$.

2. Compute sub-sample quantile τ for sub-sample b as $\hat{q}_\tau^b$, $b = 1, 2, ..., B$.

3. Compute the bootstrapped sample quantile mean as $\bar{\hat{q}}_\tau^B = \frac{1}{B}\sum_{b=1}^B \hat{q}_\tau^b, b = 1, 2, ..., B$.

4. Finally, compute bootstrapped sample quantile variance as:

$$\widehat{Var^B}(q_\tau) = \frac{1}{B}\sum_{b=1}^{B}\left(\hat{q}_\tau^b - \bar{\hat{q}}_\tau^B\right)^2, b = 1, 2, ..., B.$$

Then our proposed quantile density estimator is

$$\hat{f}_v(q_\tau) = \sqrt{\frac{\tau(1-\tau)}{n\widehat{Var^B(q_\tau)}}}. \tag{4}$$

This estimator is based on the asymptotic result stating that if $n\widehat{Var^B(q_\tau)} \xrightarrow{p} Var(q_\tau)$ as $n \to \infty$ then $\hat{f}_v(q_\tau) \xrightarrow{p} f(q_\tau)$ as $n \to \infty$. Hall and Martin (1988) derives convergence properties for quantile variance bootstrap estimation and shows consistency and the rate of convergence of the bootstrap quantile variance estimator and bootstrap sparsity function estimator.

In particular, Hall and Martin (1988, Theorem 2.2, page 262) shows that

$$n^{\frac{5}{4}}(\widehat{Var^B(q_\tau)} - Var(q_\tau)) \xrightarrow{d} N\left(0, 2\pi^{1/2}[\tau(1-\tau)]^{3/2} f(q_\tau)^{-4}\right). \tag{5}$$

Thus, as a corollary Hall and Martin (Corollary 2.2, p.263 1988) obtains the main result,

$$n^{\frac{1}{4}}\left(\hat{f}_v(q_\tau) - f(q_\tau)\right) \xrightarrow{d} N\left(0, 1/2\pi^{-1/2}[\tau(1-\tau)]^{-1/2} f(q_\tau)^2\right). \tag{6}$$

Then, the rate of convergence of $\hat{f}_v(q_\tau)$ is also of order $n^{-1/4}$, which is inferior to regular kernel rate of converge $n^{-2/5}$ (see Pagan and Ullah, 1999, for standard resultsof the kernel density estimator). As a result, this proposed method will be asymptotically inferior to the standard kernel estimator.

4 Quantile-mean covariance density estimator

The quantile-mean covariance function, $C(\hat{q}_\tau, \bar{x})$, is the asymptotic covariance between the sample quantiles, i.e. $\hat{q}_\tau$, with index $\tau \in (0, 1)$ and the sample mean, i.e., $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,

$$C(\hat{q}_\tau, \bar{x}) \equiv \lim_{n\to\infty} nCov(\hat{q}_\tau, \bar{x}). \tag{7}$$

Define the expected quantile loss function as

$$\varpi(\tau) = E[\rho_\tau(X - q_\tau)] = \tau(\mu - E[X|X < q_\tau]) = \tau\left(\mu - \frac{1}{\tau}E[1[X < q_\tau]X]\right), \tag{8}$$

where $\rho_\tau(u) = \{\tau - 1[u \leq 0]\}u$ is the quantile check function in Koenker and Bassett (1978).

By Ferguson (1999),

$$f(q_\tau) = \frac{\varpi(\tau)}{C(\hat{q}_\tau, \bar{x})}. \tag{9}$$

The estimator of $f(q_\tau)$ requires consistent estimators of $\varpi(\tau)$ and $C(\hat{q}_\tau, \bar{x})$. Consider the following estimator for $\varpi(\tau)$,

$$\hat{\varpi}(\tau) = \tau\left(\bar{x} - \frac{1}{n\tau}\sum_{i=1}^{n} X_i 1[X_i < \hat{q}_\tau]\right),$$

and note that $\hat{\varpi}(\tau) \xrightarrow{p} \varpi(\tau)$ as $n \to \infty$ (see Bera et al. (2016) for a derivation).

The key point for achieving a consistent density estimator for $f(q_\tau)$ is to consistently estimate $C(\hat{q}_\tau, \bar{x})$. That is, we want to estimate a "covariance" between two random variables (e.g., as if we would estimate the covariance between $\bar{x}$ and $\hat{\sigma}^2$ or any other two "moments").

We propose a non-parametric bootstrap estimator which takes B random sub-samples of size n out of n observations and then follows these steps:

1. Consider B bootstrap sub-samples of size n, $\{X_i^b\}_{i=1}^n, b = 1, 2, ..., B$.

2. For each τ, compute the sub-sample mean $\bar{x}^b$ and sub-sample quantile, $\hat{q}_\tau^b$, $b = 1, 2, ..., B$.

3. Compute bootstrapped sample quantile-mean covariance as:
$$\hat{C}^B(\hat{q}_\tau, \bar{x}) = \frac{1}{B}\sum_{b=1}^{B}\left(\bar{x}^b \hat{q}_\tau^b\right) - \left(\frac{1}{B}\sum_{b=1}^{B}\bar{x}^b\right)\left(\frac{1}{B}\sum_{b=1}^{B}\hat{q}_\tau^b\right).$$

Then our proposed quantile density estimator is

$$\hat{f}_c(q_\tau) = \frac{\hat{\varpi}(\tau)}{n\hat{C}^B(\hat{q}_\tau, \bar{x})}. \tag{10}$$

5 Smoothing methods

In practice, the density estimator may have several kinks in small samples because of the non-continuous nature of quantile estimators. That is, the density estimators can only be estimated at q_τ for the τs for which we can estimate a sample quantile. Then we can propose smoothing strategies to obtain smoother estimators.

Moving average. Suppose a given grid of τ values $\mathcal{T} = \{\tau_1, \tau_2, ..., \tau_T\}$ such that $\tau_i < \tau_{i+1}, i = 1, 2, ..., T-1$. For simplicity we will write below $f(\tau)$ instead of $f(q_\tau)$.

Then, consider a moving average of $2m+1$, i.e. we would consider m quantiles to the left and m quantiles to the right to take an average, for τ_i, $i = m+1, ..., m-1$. m is here a smoothing parameter and we can thus analyze the asymptotic properties with respect to n and m to get optimality properties.

$$
\begin{aligned}
\hat{f}(\tau_i)^{MA} &= \frac{1}{2m+1}\Big(\hat{f}(\tau_{i-m}) + \hat{f}(\tau_{i-m+1}) + ... \\
&\quad + \hat{f}(\tau_{i-1}) + \hat{f}(\tau_i) + \hat{f}(\tau_{i+1}) + ... + \hat{f}(\tau_{i+m})\Big).
\end{aligned}
$$

Alternatively, consider a moving average weighted by kernel function Ψ,

$$
\begin{aligned}
\hat{f}(\tau_i)^{WMA} &= \frac{1}{2m+1}\Big(\Psi(\tau_{i-m} - \tau_i)\hat{f}(\tau_{i-m}) + \Psi(\tau_{i-m+1} - \tau_i)\hat{f}(\tau_{i-m+1}) + ... \\
&\quad + \Psi(\tau_{i-1} - \tau_i)\hat{f}(\tau_{i-1}) + \Psi(0)\hat{f}(\tau_i) + \Psi(\tau_{i+1} - \tau_i)\hat{f}(\tau_{i+1}) + ... \\
&\quad + \Psi(\tau_{i+m} - \tau_i)\hat{f}(\tau_{i+m})\Big).
\end{aligned}
$$

HP filter. Hodrick and Prescott (1997) proposed a very popular method for decomposing time-series into trend and cycle, which is a well-known penalized spline smoother. We apply this as follows:

$$\hat{f}(\tau_i)^{HP} = \min_{\hat{f}(\tau_i)^{HP}} \{\sum_{i=1}^{T}(\hat{f}(\tau_i) - \hat{f}(\tau_i)^{HP})^2 + ...$$
$$+\lambda \sum_{i=1}^{T}[(\hat{f}(\tau_i)^{HP} - \hat{f}(\tau_{i-1})^{HP}) - (\hat{f}(\tau_{i-1})^{HP} - \hat{f}(\tau_{i-2})^{HP})]^2\},$$

where T is the number of quantiles estimated and λ is a smoothing parameter.

6 Monte Carlo simulations

6.1 Data generating processes

In this section we present finite sample simulations to evaluate the performance of the $\hat{f}_v$ and $\hat{f}_c$ density estimators.

For a given simulation j, the integrated squared error ISE_j is defined as $ISE_j = \int \left(\hat{f}_j(\hat{q}_\tau) - f(q_\tau)\right)^2 d\tau$ where $f(q_\tau)$ is the known density function evaluated at the true τ-quantile and $\hat{f}_j(\hat{q}_\tau)$ is the estimated density for simulation j. Then, the mean integrated squared error (MISE) is computed as $MISE = \frac{1}{M}\sum_{j=1}^{M} ISE_j$, where M is the number of simulations.

As a benchmark for each data generating process (DGP) we also estimate its density with the kernel estimator, which takes the form

$$\hat{f}_k(\hat{q}_\tau) = \frac{1}{nh}\sum_{i=1}^{n} k\left(\frac{X_i - \hat{q}_\tau}{h}\right), \tag{11}$$

where n is the sample size, h is the bandwidth, and $k(.)$ the kernel function. We use the default kernel $k(.)$ and bandwidth h set by `STATA` which are the Epanechnikov kernel and optimal Gaussian bandwidth.

In order to test the accuracy of our estimators we generate $M = 1000$ random samples from the following DGPs, where all random variables are standardized to have a variance 1: (i) standard Gaussian distribution, $N(0,1)$; (ii) symmetric Laplace distribution; (ii) asymmetric Gumbel, $Gumbel(-\gamma\frac{\sqrt{6}}{\pi}, \frac{\sqrt{6}}{\pi})$ with $\gamma \approx 0.5772$; and (iv) $Gamma(1,1)$.

As noted above, the discrete nature of quantiles produces in general a disperse grid of estimating points. In order to reduce variability and improve estimators performance we also compute smoothed versions (MA, WMA, and HP) of each estimator and evaluate their MISE. Given that quantile variance and quantile-mean covariance are very volatile at extremes of the distributions, we also explore $\hat{f}_v(q_\tau)$ and $\hat{f}_c(q_\tau)$ performance after trimming the distribution support for left and right at ±1%, ±2.5% and ±5%. Moreover, in order to avoid upper-extreme values of $\hat{f}_v(q_\tau)$ and $\hat{f}_c(q_\tau)$ coming from near-zero quantile variance and quantile-mean covariance estimations, we also explore replacing extreme density values at the top 1%, 2.5% and 5%. We did so by replacing extreme values with their two nearest neighbors' average density.

Unless otherwise specified, the simulations correspond to $M = 1000$ replications; $B = 500$ bootstraps in each case; quantiles grid given by $\tau \in \{0.01, 0.02, \ldots, 0.99\}$ (Taus:100); and sample size $N = 1000$.

6.2 Alternative distributions

Tables 1 to 4, one for each DGP, present the results of Monte Carlo simulations in terms of bias and MISE for different density estimators that make use of quantiles variance and quantile-mean covariance.[4] Furthermore, as a benchmark for comparing their performance, we also present in the last rows of each table the ratio between each estimator's MISE and that of the standard kernel density estimator, $\hat{f}_k$.

Consider the first column in tables 1 to 4, which correspond to the density estimations before trimming. We can see that estimations $\hat{f}_v(q_\tau)$ and $\hat{f}_c(q_\tau)$ of the Gaussian process in table 1 and Gumbel in table 3 perform worse than $\hat{f}_k(q_\tau)$ estimator even after smoothing. Also, for those DGPs, the best performance in terms of relative MISE is observed for the $\hat{f}_v(q_\tau)^{HP}$ estimator achieving a MISE about 50% greater than $\hat{f}_k(q_\tau)$. For the Laplace distribution presented in table 2, however, $\hat{f}_v(q_\tau)^{HP}$ MISE is only 63% of that achieved by $\hat{f}_k(q_\tau)$ with most of the reduction coming from $\hat{f}_v(q_\tau)^{HP}$'s lower bias. Furthermore, for the Gamma distribution in table 4, most of the proposed estimators over-perform the traditional kernel estimation. For instance, $\hat{f}_c(q_\tau)^{MA}$'s MISE is 94% of that resulting from $\hat{f}_k(q_\tau)$ and $\hat{f}_v(q_\tau)^{HP}$ is as little as 7.6% of $\hat{f}_k(q_\tau)$'s MISE.

In summary, the $\hat{f}_v(q_\tau)$ estimator outperforms $\hat{f}_c(q_\tau)$ for most of the

[4] Four based on quantile variance ($\hat{f}_v(q_\tau)$, $\hat{f}_v(q_\tau)^{MA}$, $\hat{f}_v(q_\tau)^{WMA}$, and $\hat{f}_v(q_\tau)^{HP}$), four based on quantile-mean covariance ($\hat{f}_c(q_\tau)$, $\hat{f}_c(q_\tau)^{MA}$, $\hat{f}_c(q_\tau)^{WMA}$, and $\hat{f}_c(q_\tau)^{HP}$)

DGPs, and they are better than $\hat{f}_k(q_\tau)$'s for some of the proposed distributions like Laplace and Gamma.

Due to the bootstrap sub-sampling nature, $\hat{f}_v(q_\tau)$ and $\hat{f}_c(q_\tau)$ estimators are very unstable at the extremes of the distribution support. A major risk for stability arises from sporadic sub-samples with quantile-variance or quantile-mean covariance near-0 (i.e., $\widehat{V(q_\tau)} \approx 0$ or $\widehat{C(\hat{q}_\tau, \bar{x})} \approx 0$). This is why, we also evaluate the density estimators' MISE after trimming the DGP support. These results appear in the three "Support Trim" columns of tables 1 to 4. We find that just by trimming the lowest and highest percentiles (i.e. $\pm 1\%$ of support), the proposed density estimators improves performance notoriously and that even before smoothing $\hat{f}_c(q_\tau)$ and $\hat{f}_v(q_\tau)$ override kernel estimation for most of the distributions. For instance, for the Gaussian distribution in table 1, $\hat{f}_c(q_\tau)^{HP}$ performs two times better than $\hat{f}_k(q_\tau)$, while $\hat{f}_v(q_\tau)^{HP}$ performs four times better. Moreover, after $\pm 1\%$ trimming, all the smoothed estimators perform better than $\hat{f}_k(q_\tau)$ for the four distributions.

Finally, we explore the proposed estimators' performance after replacing the upper-extreme values. Firstly, we estimate the density as before using all the information available. Secondly, we detect upper-extreme values of point estimation of the density estimator, and we replace these extreme values for a lineal interpolation between the nearest density points. This process has the advantage that it does not affect the support and produce a complete density estimation (i.e., without trimming). We find that replacing 1% of upper-extreme values for a lineal interpolation has slightly better results in terms of MISE than cutting extremes of support in ±1%.

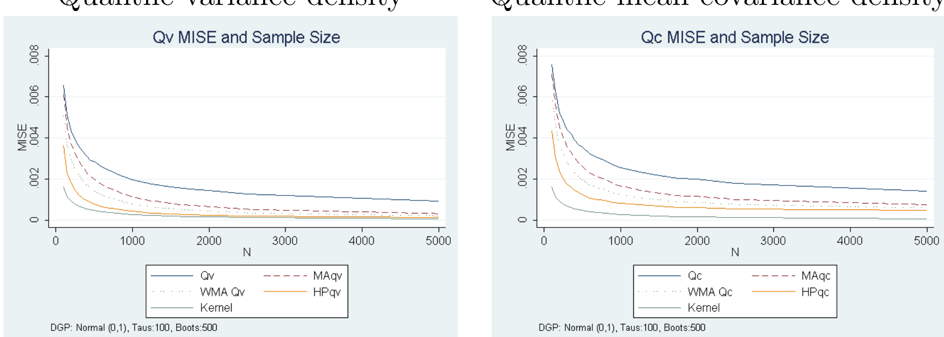
6.3 Alternative sample sizes, number of bootstrap and number of quantiles

Figures 1 to 3 plot resulting MISE from the Monte Carlo simulations for a Normal DGP without trimming or replacing extreme values.

In figure 1 we show how MISE of $\hat{f}_v(q_\tau)$ and $\hat{f}_c(q_\tau)$ decreases as the sample size grows. We can also see that bootstrap estimators are dominated by kernel estimation, confirming Hall and Martin (1988)'s theoretical result, which states that the kernel density estimator has a faster rate of convergence. However, we can also see that $\hat{f}_v(q_\tau)^{HP}$ with sample size greater than 1200 performs almost identically to kernel.

Figure 1: MISE and sample size

Quantile variance density | Quantile-mean covariance density

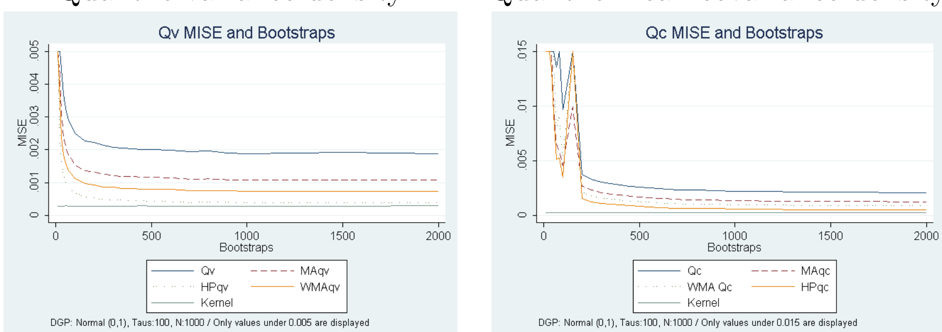


Notes: Number of Monte Carlo simulations is 1000. Bootstraps: 500. Taus: 100. Sample size: Discrete grid over the set {10, 20, 30, 40, 50, 65, 80, ,100, 150, 200, 250, 300, 350, 400, 450, 500, 600, 700, 800, 900, 1000, 1500, 2000, 2500, 5000}.

Figure 2 evaluates $\hat{f}_c(q_\tau)$ and $\hat{f}_v(q_\tau)$ performance when increasing the number of bootstrap replications, leaving the sample size and remaining parameters fixed. From this graph we can conclude that $\hat{f}_v(q_\tau)$ requires less bootstrap replications than $\hat{f}_c(q_\tau)$ to become stable, and that increasing the number of bootstraps leads to convergence.

Figure 2: MISE and number of boostraps

Quantile variance density   Quantile-mean covariance density

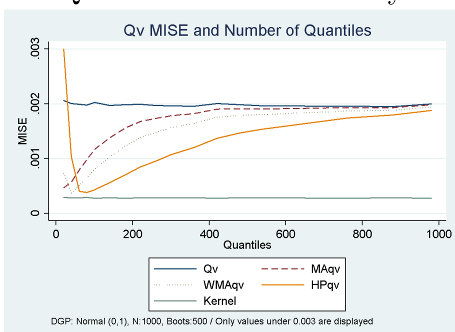


Notes: Number of Monte Carlo simulations: 1000. Sample size: 1000. Taus: 100. Bootstraps: Discrete grid over the set {100 150 200 250 300 350 400 450 500 600 700 800 900 1000 1250 1500 1750 2000 2500 5000}.

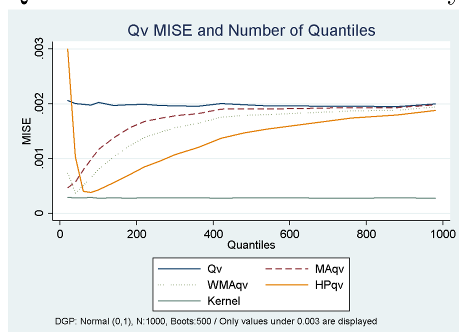
Finally, in figure 3 we evaluate how the number of quantiles used in the grid for the density estimation procedure affects the MISE. We find that for $\hat{f}_c(q_\tau)$ and $\hat{f}_v(q_\tau)$ there is no trade-off between the number of quantiles and MISE. However, there is a trade-off between quantiles and MISE for smoothed estimators. Figure 3 show that for Gaussian DGP, after some point, increasing the number of quantiles only increases the MISE. The reason for this finding is that smoothing techniques improves density estimation performance due to variance reduction. However, as the number of point estimation converges to the sample size (which is fixed at $N = 1000$), the variance of smoothed estimators also converges to the raw estimator variance.

Figure 3: MISE and number of quantiles

Quantile variance density



Quantile-mean covariance density



Notes: Number of Monte Carlo simulations: 1000. Sample size: 1000. Bootstraps: 500. Taus: Discrete grid over the set {20 40 60 80 100 140 180 240 320 480 540 760 880 980} for number of quantiles.

7 Summary and conclusions

We evaluate two density estimators, which respectively make use of the bootstrapped quantile variance and quantile-mean covariance for non-parametrically estimating the density function of a continuous random variable. In sum, given the results of the simulations we believe that this is a promising strategy. Several advances could be done for improving the estimators' performance. For instance, by combining them with improved techniques for quantile variance and covariance estimations such as m-out-of-n bootstrap and exact bootstrap. We hope that this work will encourage more research on this topic.

Our proposed approach for density estimation is flexible enough to be implemented, and presumably improved, by any overperfoming method for variance and covariance estimation such as those proposed by Sheater (1986), Huang (1991), Janas (1993), Rao et al. (1997), Hutson and Ernst (2000), Ho and Lee (2005), Cheung and Lee (2005), and Alin et al. (2017).

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Table 1: **Gaussian Distribution - Bias and Mean Integrated Square Error**

		Support Trim			Range Upper-Trim		
		± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
$\hat{f}_v(q_\tau)$	-0.0039	-0.0039	-0.0040	-0.0041	-0.0036	-0.0027	-0.0015
$\hat{f}_v(q_\tau)^{MA}$	-0.0032	-0.0033	-0.0034	-0.0035	-0.0030	-0.0019	-0.0007
$\hat{f}_v(q_\tau)^{WMA}$	-0.0023	-0.0025	-0.0028	-0.0034	-0.0020	-0.0011	0.0001
$\hat{f}_v(q_\tau)^{HP}$	-0.0039	-0.0039	-0.0040	-0.0041	-0.0036	-0.0027	-0.0015
$\hat{f}_c(q_\tau)$	-0.0080	-0.0081	-0.0082	-0.0084	-0.0077	-0.0067	-0.0055
$\hat{f}_c(q_\tau)^{MA}$	-0.0075	-0.0076	-0.0077	-0.0080	-0.0072	-0.0061	-0.0048
$\hat{f}_c(q_\tau)^{WMA}$	-0.0064	-0.0066	-0.0070	-0.0075	-0.0061	-0.0051	-0.0039
$\hat{f}_c(q_\tau)^{HP}$	-0.0080	-0.0081	-0.0082	-0.0084	-0.0077	-0.0067	-0.0055
$\hat{f}_k(q_\tau)$	0.0024	0.0079	0.0087	0.0098	0.0075	0.0074	0.0073
MISE							
$\hat{f}_v(q_\tau)$	0.0020	0.0020	0.0021	0.0022	0.0019	0.0018	0.0016
$\hat{f}_v(q_\tau)^{MA}$	0.0008	0.0012	0.0012	0.0012	0.0011	0.0010	0.0009
$\hat{f}_v(q_\tau)^{WMA}$	0.0011	0.0008	0.0009	0.0013	0.0008	0.0007	0.0007
$\hat{f}_v(q_\tau)^{HP}$	0.0004	0.0004	0.0004	0.0005	0.0004	0.0004	0.0004
$\hat{f}_c(q_\tau)$	0.0026	0.0026	0.0027	0.0028	0.0025	0.0023	0.0021
$\hat{f}_c(q_\tau)^{MA}$	0.0017	0.0017	0.0017	0.0018	0.0016	0.0015	0.0014
$\hat{f}_c(q_\tau)^{WMA}$	0.0012	0.0013	0.0013	0.0018	0.0012	0.0011	0.0010
$\hat{f}_c(q_\tau)^{HP}$	0.0008	0.0008	0.0009	0.0009	0.0008	0.0008	0.0007
$\hat{f}_k(q_\tau)$	0.0003	0.0017	0.0017	0.0019	0.0016	0.0016	0.0016
Ratio MISE							
$\hat{f}_v(q_\tau)$	7.0496	1.2088	1.1946	1.1506	1.1925	1.0864	0.9919
$\hat{f}_v(q_\tau)^{MA}$	2.7991	0.6940	0.6865	0.6626	0.6875	0.6299	0.5758
$\hat{f}_v(q_\tau)^{WMA}$	4.0469	0.4760	0.4905	0.7041	0.4765	0.4401	0.4061
$\hat{f}_v(q_\tau)^{HP}$	1.5128	0.2584	0.2561	0.2502	0.2584	0.2424	0.2284
$\hat{f}_c(q_\tau)$	9.1981	1.5768	1.5579	1.5000	1.5568	1.4225	1.2984
$\hat{f}_c(q_\tau)^{MA}$	5.9198	1.0149	1.0032	0.9669	1.0065	0.9258	0.8462
$\hat{f}_c(q_\tau)^{WMA}$	4.4111	0.7491	0.7582	0.9794	0.7515	0.6960	0.6405
$\hat{f}_c(q_\tau)^{HP}$	2.8892	0.4956	0.4922	0.4807	0.4938	0.4633	0.4324

Notes: The simulations correspond to $M = 1000$; $B = 500$ bootstraps; quantiles grid given by $\tau \in \{0.01, 0.02, \ldots, 0.99\}$; and sample size $N = 1000$. $\hat{f}_v(q_\tau)$: Quantile Variance Density. $\hat{f}_c(q_\tau)$: Quantile-Mean Covariance Density. $\hat{f}_k(q_\tau)$: Kernel density estimation. $\hat{f}_v(q_\tau)^{MA}$ and $\hat{f}_c(q_\tau)^{MA}$ are Moving Average smoothing. $\hat{f}_v(q_\tau)^{WMA}$ and $\hat{f}_c(q_\tau)^{WMA}$ are Kernel-Weighted Moving Average ($bw = 3$) smoothing. $\hat{f}_v(q_\tau)^{HP}$ and $\hat{f}_c(q_\tau)^{HP}$ are HP filter ($\lambda = 1600$) smoothing. Ratio MISE is the ratio of each density estimator's MISE over $\hat{f}_k(q_\tau)$.

Table 2: **Laplace Distribution - Bias and Mean Integrated Square Error**

		Support Trim			Range Upper-Trim		
		± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
$\hat{f}_v(q_\tau)$	-0.0040	-0.0041	-0.0042	-0.0043	-0.0037	-0.0022	-0.0004
$\hat{f}_v(q_\tau)^{MA}$	-0.0036	-0.0037	-0.0037	-0.0039	-0.0033	-0.0017	0.0003
$\hat{f}_v(q_\tau)^{WMA}$	-0.0027	-0.0027	-0.0027	-0.0026	-0.0024	-0.0009	0.0010
$\hat{f}_v(q_\tau)^{HP}$	-0.0040	-0.0041	-0.0042	-0.0043	-0.0037	-0.0022	-0.0004
$\hat{f}_c(q_\tau)$	-0.0089	-0.0091	-0.0093	-0.0097	-0.0085	-0.0069	-0.0048
$\hat{f}_c(q_\tau)^{MA}$	-0.0088	-0.0089	-0.0091	-0.0094	-0.0083	-0.0066	-0.0044
$\hat{f}_c(q_\tau)^{WMA}$	-0.0076	-0.0076	-0.0076	-0.0075	-0.0072	-0.0055	-0.0035
$\hat{f}_c(q_\tau)^{HP}$	-0.0089	-0.0091	-0.0093	-0.0097	-0.0085	-0.0069	-0.0048
$\hat{f}_k(q_\tau)$	0.0060	0.0151	0.0164	0.0173	0.0141	0.0121	0.0100
MISE							
$\hat{f}_v(q_\tau)$	0.0034	0.0035	0.0036	0.0038	0.0033	0.0030	0.0028
$\hat{f}_v(q_\tau)^{MA}\hat{f}_c(q_\tau)^{WMA}$	0.0014	0.0021	0.0021	0.0023	0.0020	0.0019	0.0018
$\hat{f}_v(q_\tau)^{WMA}$	0.0020	0.0014	0.0015	0.0017	0.0014	0.0014	0.0014
$\hat{f}_v(q_\tau)^{HP}$	0.0009	0.0009	0.0009	0.0010	0.0009	0.0009	0.0010
$\hat{f}_c(q_\tau)$	0.0045	0.0046	0.0048	0.0050	0.0044	0.0040	0.0037
$\hat{f}_c(q_\tau)^{MA}$	0.0030	0.0031	0.0032	0.0034	0.0030	0.0028	0.0027
$\hat{f}_c(q_\tau)^{WMA}$	0.0023	0.0023	0.0023	0.0026	0.0022	0.0021	0.0021
$\hat{f}_c(q_\tau)^{HP}$	0.0016	0.0016	0.0017	0.0018	0.0016	0.0016	0.0016
$\hat{f}_k(q_\tau)$	0.0014	0.0057	0.0061	0.0065	0.0055	0.0053	0.0050
Ratio MISE							
$\hat{f}_v(q_\tau)^{QV}$	2.3960	0.6025	0.5872	0.5747	0.6018	0.5765	0.5678
$\hat{f}_v(q_\tau)^{MA}$	1.0043	0.3607	0.3521	0.3455	0.3634	0.3585	0.3643
$\hat{f}_v(q_\tau)^{WMA}$	1.4330	0.2481	0.2387	0.2604	0.2560	0.2589	0.2711
$\hat{f}_v(q_\tau)^{HP}$	0.6284	0.1583	0.1550	0.1532	0.1619	0.1723	0.1901
$\hat{f}_c(q_\tau)$	3.1905	0.8024	0.7820	0.7655	0.8002	0.7656	0.7468
$\hat{f}_c(q_\tau)^{MA}$	2.1379	0.5381	0.5252	0.5152	0.5412	0.5312	0.5298
$\hat{f}_c(q_\tau)^{WMA}$	1.6086	0.3971	0.3801	0.3958	0.4090	0.4085	0.4156
$\hat{f}_c(q_\tau)^{HP}$	1.1286	0.2844	0.2784	0.2750	0.2890	0.2981	0.3133

See notes to Table 1.

Table 3: **Gumbel Distribution - Bias and Mean Integrated Square Error**

		Support Trim			Range Upper-Trim		
		± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
$\hat{f}_v(q_\tau)$	-0.0043	-0.0044	-0.0044	-0.0045	-0.0040	-0.0028	-0.0016
$\hat{f}_v(q_\tau)^{MA}$	-0.0035	-0.0036	-0.0037	-0.0039	-0.0032	-0.0020	-0.0006
$\hat{f}_v(q_\tau)^{WMA}$	-0.0024	-0.0028	-0.0031	-0.0039	-0.0021	-0.0010	0.0003
$\hat{f}_v(q_\tau)^{HP}$	-0.0043	-0.0044	-0.0044	-0.0045	-0.0040	-0.0028	-0.0016
$\hat{f}_c(q_\tau)$	-0.0092	-0.0093	-0.0094	-0.0096	-0.0089	-0.0076	-0.0062
$\hat{f}_c(q_\tau)^{MA}$	-0.0085	-0.0086	-0.0088	-0.0090	-0.0081	-0.0068	-0.0053
$\hat{f}_c(q_\tau)^{WMA}$	-0.0073	-0.0076	-0.0080	-0.0088	-0.0070	.0057	-0.0043
$\hat{f}_c(q_\tau)^{HP}$	-0.0092	-0.0093	-0.0094	-0.0096	-0.0089	-0.0076	-0.0062
$\hat{f}_k(q_\tau)^{HP}$	0.0042	0.0119	0.0131	0.0145	0.0113	0.0110	0.0107
MISE							
$\hat{f}_v(q_\tau)$	0.0027	0.0027	0.0028	0.0029	0.0026	0.0023	0.0022
$\hat{f}_v(q_\tau)^{MA}$	0.0011	0.0015	0.0016	0.0016	0.0015	0.0013	0.0012
$\hat{f}_v(q_\tau)^{WMA}$	0.0015	0.0011	0.0011	0.0020	0.0010	0.0009	0.0009
$\hat{f}_v(q_\tau)^{HP}$	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005
$\hat{f}_c(q_\tau)$	0.0036	0.0037	0.0038	0.0039	0.0035	0.0032	0.0030
$\hat{f}_c(q_\tau)^{MA}$	0.0024	0.0024	0.0025	0.0025	0.0023	0.0021	0.0019
$\hat{f}_c(q_\tau)^{WMA}$	0.0018	0.0018	0.0019	0.0030	0.0018	0.0016	0.0015
$\hat{f}_c(q_\tau)^{HP}$	0.0012	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011
$\hat{f}_k(q_\tau)$	0.0004	0.0024	0.0024	0.0026	0.0023	0.0023	0.0023
Ratio MISE							
$\hat{f}_v(q_\tau)$	6.7457	1.1506	1.1447	1.0959	1.1360	1.0252	0.9383
$\hat{f}_v(q_\tau)^{MA}$	2.6628	0.6535	0.6508	0.6236	0.6486	0.5891	0.5407
$\hat{f}_v(q_\tau)^{WMA}$	3.8317	0.4533	0.4627	0.7634	0.4518	0.4142	0.3842
$\hat{f}_v(q_\tau)^{HP}$	1.5397	0.2573	0.2537	0.2442	0.2628	0.2474	0.2357
$\hat{f}_c(q_\tau)$	9.2464	1.5737	1.5641	1.4913	1.5560	1.4073	1.2871
$\hat{f}_c(q_\tau)^{MA}$	5.9742	1.0173	1.0112	0.9626	1.0110	0.9221	0.8461
$\hat{f}_c(q_\tau)^{WMA}$	4.5464	0.7726	0.7796	1.1312	0.7713	0.7097	0.6567
$\hat{f}_c(q_\tau)^{HP}$	3.1642	0.5349	0.5313	0.5110	0.5395	0.5066	0.4777

See notes to Table 1.

Table 4: **Gamma Distribution - Bias and Mean Integrated Square Error**

		Support Trim			Range Upper-Trim		
		± 0.01	± 0.025	± 0.05	-0.01	-0.025	-0.05
Bias							
$\hat{f}_v(q_\tau)$	-0.0076	-0.0071	-0.0068	-0.0066	-0.0054	-0.0018	0.0014
$\hat{f}_v(q_\tau)^{MA}$	-0.0068	-0.0067	-0.0067	-0.0065	-0.0059	-0.0034	-0.0009
$\hat{f}_v(q_\tau)^{WMA}$	-0.0079	-0.0075	-0.0072	-0.0056	-0.0053	-0.0010	0.0026
$\hat{f}_v(q_\tau)^{HP}$	-0.0076	-0.0071	-0.0068	-0.0066	-0.0054	-0.0018	0.0014
$\hat{f}_c(q_\tau)$	-0.0209	-0.0205	-0.0186	-0.0163	-0.0088	0.0010	0.0070
$\hat{f}_c(q_\tau)^{MA}$	-0.0181	-0.0173	-0.0167	-0.0154	-0.0146	-0.0092	-0.0050
$\hat{f}_c(q_\tau)^{WMA}$	-0.0216	-0.0236	-0.0214	-0.0184	-0.0061	0.0052	0.0116
$\hat{f}_c(q_\tau)^{HP}$	-0.0209	-0.0205	-0.0186	-0.0163	-0.0088	0.0010	0.0070
$\hat{f}_k(q_\tau)$	0.0658	0.0938	0.0923	0.0792	0.0944	0.0870	0.0805
MISE							
$\hat{f}_v(q_\tau)$	0.0103	0.0093	0.0088	0.0078	0.0089	0.0076	0.0070
$\hat{f}_v(q_\tau)^{MA}$	0.0039	0.0039	0.0039	0.0037	0.0039	0.0036	0.0034
$\hat{f}_v(q_\tau)^{WMA}$	0.0040	0.0044	0.0054	0.0175	0.0037	0.0037	0.0040
$\hat{f}_v(q_\tau)^{HP}$	0.0020	0.0019	0.0019	0.0018	0.0018	0.0018	0.0019
$\hat{f}_c(q_\tau)$	0.6270	0.0382	0.0198	0.0135	0.5456	0.7360	0.7576
$\hat{f}_c(q_\tau)^{MA}$	0.0242	0.0103	0.0090	0.0075	0.0304	0.0737	0.0777
$\hat{f}_c(q_\tau)^{WMA}$	0.3119	0.0404	0.0231	0.0329	0.4029	0.7122	0.7529
$\hat{f}_c(q_\tau)^{HP}$	0.1034	0.0137	0.0085	0.0060	0.1381	0.2940	0.3123
$\hat{f}_k(q_\tau)$	0.0258	0.0412	0.0397	0.0324	0.0418	0.0381	0.0351
Ratio MISE							
$\hat{f}_v(q_\tau)$	0.4014	0.2261	0.2216	0.2403	0.2136	0.2006	0.1990
$\hat{f}_v(q_\tau)^{MA}$	0.1526	0.0957	0.0989	0.1145	0.0924	0.0941	0.0976
$\hat{f}_v(q_\tau)^{WMA}$	0.1556	0.1074	0.1358	0.5407	0.0886	0.0961	0.1131
$\hat{f}_v(q_\tau)^{HP}$	0.0768	0.0461	0.0473	0.0565	0.0434	0.0464	0.0530
$\hat{f}_c(q_\tau)$	24.3396	0.9267	0.4981	0.4174	13.0656	19.3351	21.5978
$\hat{f}_c(q_\tau)^{MA}$	0.9394	0.2501	0.2273	0.2333	0.7279	1.9367	2.2148
$\hat{f}_c(q_\tau)^{WMA}$	12.1093	0.9813	0.5811	1.0152	9.6493	18.7112	21.4627
$\hat{f}_c(q_\tau)^{HP}$	4.0142	0.3329	0.2151	0.1839	3.3071	7.7230	8.9031

See notes to Table 1.