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Memory, multiple equilibria and emerging market crises

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Memory, multiple equilibria and emerging market crises

Financial crises
Sudden stops
Small open economies
Ergodicity
Recursive equilibrium
Generalized markov equilibria

We present a new Generalized Markov Equilibrium (GME) approach to studying sudden stops and financial crises in emerging countries in the canonical small open economy model with equilib-rium price-dependent collateral constraints. Our approach to characterizing and computing stochastic equilibrium dynamics is global, encompasses recursive equilibrium as a special case, yet allows for a much more flexible approach to modeling memory in such models that are known to have multiple equilibrium. We prove the existence of ergodic GME selections from the set of sequential competitive equilibrium, and show that at the same time ergodic GME selectors can replicate all the observed phases of the macro crises associated with a sudden stop (boom, collapse, spiralized recession, recov-ery) while still being able to capture the long-run stylized behavior of the data. We also compute stochastic equilibrium dynamics associated with stationary and nonstationary GME selections, and we find that: a) the ergodic GME selectors generate stochastic dynamics that are less financially constrained with respect to stationary non-ergodic paths; and, b) non-stationary GME selections ex-hibit a great range of fluctuations in macroeconomic aggregates compared to the stationary selections. From a theoretical perspective, we prove the existence of both sequential competitive equilibrium and (minimal state space) recursive equilibrium, as well as provide a complete theory of robust recursive equilibrium comparative statics in deep parameters. Consistent with recent results in the literature, relative to the set of recursive equilibrium, we find 2 stationary equilibrium: one with high/over borrowing, the other with low/under borrowing. These equilibrium are extremal and "self-fulfilling" under rational expectations. The selection among these equilibria depend on observable variables and not on sunspots.

Memoria, equilibrios múltiples y crisis en países emergentes

Crisis financieras
Economía pequeña y abierta
Ergodicidad
Equilibrios recursivos

Presentamos un nuevo equilibrio generalizado de Markov (GME) para estudiar crisis de balanza de pagos en economías emergentes que sufren fricciones financieras en la forma de restricciones de colateral. Nuestro enfoque permite caracterizar y computar los equilibrios dinámicos y estocásticos en forma global, comprende a otros equilibrios recursivos (como los de espacio de esta mínimo) y representa una forma flexible de modelar "memoria" en estas economías que suelen tener equilibrios múltiples. Probamos la existencia de un GME ergódico cómo una selección del equilibrio secuencial el cual a su vez puede replicar tanto todas las fases de una crisis de balanza de pagos cómo los hechos estilizados de largo plazo. Computamos el equilibrio ergódico, estacionario y no estacionario. Encontramos que el equilibrio ergódico tiene trayectorias del consumo más suaves y que el no estacionario puede replicar un gran rango de crisis de balanza de pagos. Desde una perspectiva teórica, probamos la existencia del equilibrio secuencial y recursivo en espacio de estados mínimos, como así también resultados de estática comparativa robusta. En línea con la literatura, encontramos 2 tipos de equilibrios estacionarios: uno con alto y el otro con bajo endeudamiento. Estos equilibrios son auto-validantes en expectativas racionales y no requieren de manchas solares para su coordinación.

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Índice

1. Introduction

This paper proposes a new Generalized Markov Equilibrium (GME) approach for characterizing the stochastic equilibrium dynamics in the canonical two-sector sudden-stops model of financial crises in small open economies. We show that GME approaches to these models are both tractable and powerful, and can construct representations of the stochastic dynamics derived from minimal state space recursive equilibrium (RE) as studied in Bianchi ([12]) as well as the sequential competitive equilibrium dynamics as in Schmitt-Grohé and Uribe ([69], [70], [71]) in a single *unified* methodological setting.

More specifically, we use GME representations to build a theory of sequential competitive equilibrium (SCE) selection that induce *ergodic stationary equilibrium* in this class of models. Interesting, the very existence of binding price-dependent collateral constraints (a critical feature of sudden stops models) turns out to play a central role in obtaining our new ergodicity results. The existence of an ergodic stationary equilibrium is particularly surprising as it is well-known that the canonical sudden stops model can possess *multiple sequential competitive equilibria,* and hence stochastic equilibrium dynamics can easily become *discontinuous from initial conditions* in such models with equilibrium multiplicities. Therefore, characterizing SCE selections that induce ergodic equilibria of these models is very challenging endeavor.

An additional new feature of this paper is that we study the critical role played of *memory* in the representation of equilibrium stochastic processes when characterizing the structure of ergodic equilibrium.[1] For example, after using GME methods to prove the existence of ergodic equilibrium selectors from the set of SCE, we also show such ergodic GME selectors *cannot* be guaranteed to exist if we restrict attention to *only RE selectors* from the set of GME/SCE. Rather, what is generally needed for the existence of ergodic GME selections is *long-memory* representations of equilibrium stochastic processes, which requires one to have access to more general set of SCE than the set of (short-memory) RE.[2] Our ergodicity results allow us to replicate the observed stochastic dynamics of a sudden stops without relying on a large unanticipated shocks that impose the loss of access to capital markets *by assumption* (as it is often appealed to in the literature). Moreover, the ergodic GME selectors can be show to replicate the anatomy of different aspects of financial crises and allows us to study the effects of recurrent crises on the *long-run* behavior of macroeconomic aggregates. Thus, our approach is able to keep track of both *short and long-run dynamics.* In a sense, ergodicity of SCE selectors becomes a natural equilibrium selection device for characterizing short and long run stochastic dynamics among multiple SCE paths.

We also show GME methods can be used to distinguish between stationary, non-stationary, and ergodic equilibrium selections (and their associated long-run stochastic properties). When doing this, we find that ergodic selections have a different equilibrium stochastic structure that other GME selectors: they are less financially constrained, have smoother consumption paths, and the "size" of the recession associated with a financial crises is shown to depend on the structure of the *pre-phase* of a sudden stop. This last fact connects the role of memory associated with the ergodic stochastic process with its *short-run* predictions.

Therefore, the paper argues that for the canonical sudden stops model in the literature, GME approaches to characterizing SCE provide a new and flexible approach to the global characterization of stochastic equilibrium dynamics even in the presence of multiple equilibria, as well as provide a powerful set of tailor-made tools that allows one to extend these ideas of stochastic stability found in many important dynastic stochastic equilibrium models *without* equilibrium frictions to models where equilibrium frictions play both a critical qualitative and quantitative role in the theory. In this sense, therefore, this paper address many of the interesting questions raised in recent work that discussed the critical difference between local versus global methods for these models relative to solving equilibrium functional

[1] Although our focus in this paper is on GME methods and their ability to characterize stochastic equilibria is in the canonical model of sudden stops in the literature, as will be clear in the sequel, the central methodological theme of this paper is much more general, and makes the case for GME approaches in many stochastic equilibrium models with (endogenous) equilibrium collateral constraints and multiple equilibria.

[2] RE are "short memory" in the sense they are Markovian equilibrium that depend on only the current set of minimal states (and hence, 0-memory SCE). GME representations of SCE are recursive in general on the minimal state space only if one enlarges the set of state variables to include endogenous "states" such as envelope theorems, value functions, etc. For infinite horizon economies like those in this paper, GME representations of SCE could in principle have *infinite* memory when viewed from the perspective of a minimal state space.

equations (e. g., as discussed in the new work of De Groot, Durdu and Mendoza ([25]) and Mendoza and Villallazo ([50])), but also focuses on the implications of GME methods for characterizing, computing and simulating ergodic, stationary and non-stationary equilibria.

Of independent theoretical interest, the paper proposes a new multistep monotone-map method that provides a *complete constructive qualitative theory of RE.* This multi-step approach verifies not only the existence of a nonempty complete lattice of RE with least and greatest RE, but show in what formal sense the *set* of RE exhibiting robust equilibrium comparative statics in the deep parameters of the economy. Here, the "least" RE (resp.,"greatest " RE) have nice economic interpretations relative to the recent work of Schmitt-Grohé and Uribe ([70]) as it corresponds with the low consumption/under-borrowing (resp., high consumption/over-borrowing) equilibrium discussed in that paper. Our multi-step approach is very natural, and motivated by the existence of collateral constraints, which generates a partition of the RE over states into "two equilibrium regimes " where in some states RE are collateral constrained and in the other states they are not.

To obtain monotonicity for our RE operator, we identify a new *equilibrium single-crossing* condition that naturally arises in models with "pecuniary externalities " in states where price-dependent equilibrium collateral constraints actually bind. We refer to this new form of equilibrium complementarity as a *pecuniary complementarity*, and we show it provides a global source of multiple RE. Our approach to constructing RE also allows us to provide a formal characterization of the "self-fulfilling " nature of multiple equilibrium for stochastic small open economies with *equilibrium price-dependent* collateral constraints as has been discussed recently in a series of recent papers (e. g., Schmitt-Grohé and Uribe ([70], [71]), Bianchi and Mendoza ([16]). Our characterization shows that this problem is not a form of equilibrium indeterminacy, and does not rely in any way necessarily on the structure of "sunspot equilibria "Ṙather, a central mechanism self-fulfilling multiplicities of RE is very simple: if agents believe the aggregate laws of motion of equilibrium states governing future per-capita aggregate consumption/borrowing will be low (resp., high) in the future, RE collateral constraints will be *ordered* pointwise and tighter (resp., weaker) for households because prices of nontradeable endowments will be lower (resp, higher). These expectations then can be *self-generating*, which in turn implies that least (resp., greatest) borrowing RE consumption/debt will be lower (resp., higher). Therefore, in the end, our results imply that in a least (resp., greatest) RE, equilibrium have equilibrium collateral constraint be ordered. .

1.1 Preview of our Results

The paper considers the canonical workhorse small open economy two sector model that has been studied extensively in literature over the last decade (e. g., see Bianchi ([12]) and Schmitt-Grohé and Uribe ([70], [71])). We first provide a set of general existence results for both SCE and RE for this class of models, which we believe are the first such theoretical results on sufficient conditions for existence of equilibria in this literature. These existence results are not of purely theoretical interest in this paper; rather, they are necessary to guarantee that the GME methods we develop in the paper for characterizing stochastic equilibrium dynamics are well-defined globally from all admissible initial conditions of the stochastic model.

Next, we prove the existence of GME representations for the *set* of SCE, where the GME are defined over an enlarged state space, and we provide sufficient conditions for the existence of ergodic GME selectors from the set of SCE. This allows us to introduce the role of modeling memory when representing the stochastic dynamic equilibrium paths as GME, and distinguish SCE that have different stochastic equilibrium properties relative to their stationary equilibrium (e. g., as ergodic versus invariant measures, etc). For GME paths, we prove stochastic equilibrium dynamics hit the collateral constraint in finite time with probability one starting from any initial condition. We then propose a new systematic way of forming *ergodic* GME selections for the set of SCE. These selections require long-memory structure in general to guarantee that SCE have positive "hit" rates of the collateral constraint in a probabilistic sense in finite time. Our ergodicity results have profound implications also for empirical work which seeks to estimate Sudden Stops models from a structural perspective (e. g., see the recent work on Benigno, Chen, Otrok, and Rebucci ([11])), and should be of independent interest for the econometric study of these models.

The key ingredient to identifying ergodic GME selections turns out to the occasionally binding price-dependent collateral constraint, a critical feature of all sudden stops models, that is needed to generate observed financial crises. The equilibrium collateral constraint allows us to characterize the regeneration properties of stochastic equilibrium dynamics, a critical ingredient in proving the existence of ergodic GME selectors. Mathematically, ergodic GME selectors are identified by constructing a so-called "atom" of the equilibrium Markov process (which, in particular, regenerates at the atom), a property that turns out to be deeply connected with ergodicity of stochastic equilibrium dynamics. We show how such an identification of an atom in these models relies upon the equilibrium collateral constraint itself.

We then show ergodic GME selectors have important quantitative properties when representing stochastically the anatomy of important properties of financial crises. If we say that every time the process hits the collateral constraints the model predicts a crises, our GME approach can integrate the *frequency* of them with the long-run characteristics of macroeconomic aggregates to discipline the param-eters in the model, while at the same time keep track of the short run anatomy of different types of crises. In the end, we argue that ergodicity is a type of sequential equilibrium selection criterion which selects subsets for SCE from among multiple possible (non-ergodic) GME equilibria paths. In particular, ergodic GME are able numerically to replicate observed frequencies and structural characteristics of the crises within the context of modeling multiple *long run stylized facts* associated with macroeconomic aggre-gates, whole also characterizing *short-run* stochastic features of these models related to the coexistence of Fisherian deflation/recessions dynamics in the data alongside with long run empirical regularities.

The intuition behind our GME approach is direct, and based in the existence of two stochastic equilibrium regimes, one defined when the collateral constraint does not bind, the other when this constraint is active, making the equilibrium collateral constraint the key building block to obtain any tractable approach to characterize the nature of stochastic equilibrium dynamics. Importantly, our methods are robust to the presence of multiple equilibria and the existence of discontinuous GME selections, and are able to characterize the ergodic / long-run stochastic properties of SCE by constructing a GME path that "jumps" to the atom every time the collateral constraint is "hit" and generates a crisis. Thus, the presence of multiple equilibria allows us to better match long run empirical regularities.

Using our methods, we are able connect the existence of an invariant measure (or ergodic measure) with the frequency of financial crises. Let τ_α be the time when the process hits the collateral constraint, α is the atom and $\mu(A)$ the cumulative probability of hitting A avoiding a crises. Therefore, *the frequency of a Sudden Stops affects the stationary distribution* μ. Moreover, frequent crises imply more volatility even in the steady state of the model. To understand the intuition behind this last relationship, note that every time the process hits the collateral constraint it reverts to α. The definition of an atom implies that $P_\varphi(\alpha, A) = \nu(A)$, where P is a Markovian transition kernel and φ a selection from the GME correspondence. This result implies that the realization of the process after hitting the collateral constraint, $z_{\tau_\alpha+1}$, is independent of the past, which implies that it loses all the inertia inherited from the Markovian structure of the process. This fact in turn implies that the equilibrium stochastic dynamics behaves unconditionally with respect to the past, increasing its variability.

To relate these facts with the empirical performance of the model, we need a Law of Large numbers. If the measure is ergodic, it is well known that $\sum z_t \rightarrow E_\mu(z)$ almost everywhere. Thus, the historical time periods without a Sudden Stops are allowed to shape the long run distribution of the model, affecting its ability to replicate stylized facts. Then, use ergodic GME selections, we solve the model for a parameter set borrowed from the empirical literature, and we show that the ergodic GME selections are capable of simulating a "Fisherian deflation" (i.e. a path of real exchange rate depreciation coupled with falling consumption) and a sudden stop without relying on a large, unanticipated shock that impose the loss of access to capital markets by assumption. This shock is typically represented by a sudden change in a (non-stochastic) parameter or a change in the support of the distribution of exogenous shocks.

A central theme of the paper that is new to the literature is that modeling memory is critical to address the coexistence of the anatomy of the financial crisis with the long-run structure of macroeconomic data. That is, neither of the mentioned assumptions are required once we change the type of equilibrium from the short-memory structure of RE to the potential longer memory structure of SCE via its GME representation. We show that the order of magnitude of the "recession" associated with a Sudden Stop depends on the level of consumption reached in the pre-phase which, given the unconstrained nature of this phase and the smoothness of consumption in that regime, will depend on a sequence of positive shocks to income. The longer the sequence, the bigger the difference between the observed consumption level and those at the bottom of the ergodic distribution. We show the equilibrium stochastic process can revert to the latter once we observe a financial crises. Thus, memory matters in order to capture the quantitative properties of the sudden stop and the GME is capable of capturing it.

We also compute the effects of a change in the interest rate in the long-run properties of the model, and show that it is possible to remove the ergodic component of any selection in a GME, this selection is still time invariant and thus, stationary. We then compute the difference between simulations generated by an ergodic and a stationary GME equilibrium. We find that ergodic simulations generate *smoother* consumption paths or, equivalently, agents are *less financially constrained.* Additionally, we compute for reference the non-stationary GME, and compare the stochastic properties of these GME with ergodic GME selectors. As it is expected, this equilibrium can generate large fluctuations in macroeconomic fundamentals (i.e. current account) using standard preferences and with the same shock structure. There are at least 2 important differences between the ergodic and the stationary selections. In the latter simulations overestimate i) the elasticity of average consumption and debt with respect to the interest rate, ii) the volatility of consumption for a given interest rate, which implies that debt is less volatile.

We are also able to provide a rigorous characterization of the difference between (short-memory) RE stochastic equilibrium paths versus those stochastic equilibrium dynamics associated with GME representations of SCE, where the latter class of GME representations of SCE are allowed in general to admit "long-memory" representations of SCE. In doing so, we show that modeling memory is a useful criterion in construction GME selections. We show it is possible to construct GME selectors from the set of SCE that hit the collateral constraint along a finite time path, a critical property of GME selectors that induce ergodic stationary equilibrium. In this sense, a natural GME selection criterion from these of SCE is based upon ergodicity, as it allows us to match observed stylized facts and at the same time choose the appropriate selection from the equilibrium correspondence. Short memory paths can then be embedded into the GME, allowing us to get short term match with data. At the same time, due to the existence of long-memory SCE equilibrium, we can tie the deep parameters of the model to the long run behavior of the same observed time series. In other words: we show that there exist a selection which insures ergodicity and it is also compatible with the presence of multiple different "phases" of crises, which may have variable time spells.

Finally, we provide a qualitative theory of robust *recursive equilibrium* comparative statics relative to the *set* of RE relative to all the important deep parameters of the economy.[3]. Establishing the existence and equilibrium comparative statics results for the set of RE is important for these class of models. First, RE form an important subclass of sequential equilibria in our economies for many reasons, and there are no known results on sufficient conditions for such RE to exist. Second, much of the existing literature which apply these models actually presumes RE exists, and focuses on its computation (e. g., Bianchi ([12]). Third, by focusing on RE if they exist, one can sharpen the characterization of the structural properties of equilibrium dynamics as its nature is greatly simplified. This allows use to identify *sharply* the *global* sources of multiple equilibria in this class of models. In a very precise sense, the paper therefore provides an *global* formalization of the "self-fulfilling" nature of multiple equilibrium as discussed in Schmitt-Grohé and Uribe ([70], [71])), and we show this multiple equilibrium problem is global.

1.2 This Paper and the Existing Literature

The literature on Sudden Stops and financial crises in emerging economies has been voluminous over the last two decades. The empirical regularities of Sudden Stops that characterize Sudden Stop in emerging markets are well-documented in many papers including Mendoza ([47]) and Bianchi and Mendoza ([16]), and include the following: Sudden Stops are (a) infrequent, (b) involve reversals of international capital flows and current accounts, as well as the abrupt loss of access to international capital markets, (c) create severe economic downturns in output, consumption, and investment, where collapses in business cycle

[3]This paper considers the case discrete iid endowment shocks. Introducing Markov shocks on continuous shock spaces is a significant complication. and requires typically order continuous approaches to the existence of RE question. See Pierri and Reffett ([60]).

variables are proceeded by economic expansions, and then followed by long and protracted economic slumps, (d) typically involve contemporaneous collapses of equity prices and real exchange rates during the crisis, which are also preceded by run ups of equity and exchange rate values, and then their collapse, (e) tend to occurred during big global financial events.[4]

A large theoretical literature has emerged seeking to explain these empirical regularities. What is common to all these theoretical papers is the importance of *equilibrium* price-dependent collateral constraints in understanding the structure and genesis of the emerging market financial crises. The theoretical literature begins with the important papers by Mendoza ([46], [47]), Mendoza and Smith ([48]), Bianchi and Mendoza ([13]), and Bianchi ([12]), but has continued with many recent papers exploring the different dimensions of Sudden Stops and emerging market financial crises.[5] In these models, equilibrium collateral constraints are critical in explaining not only the mechanism by which financial crises occur (e. g., the presence of binding borrowing constraints that forces emerging economies to pay down past debts, creating a possible collapse in consumption, prices, and current accounts), but the observed co-movement in between measurements of the financial crises and important macro aggregates such as consumption, output, debt, and prices over the business cycle (e. g., in providing a Fisherian deflation narrative to understanding the emergence and structure of financial crises in emerging markets).

In this paper, we study the endowment two-sector version of the small open economy model of Sudden Stops proposed in Bianchi ([12]). This model has been studied also extensively in a series of recent papers also by Schmitt-Grohé and Uribe ([69], [70], [71]), among others, where the issues of equilibrium multiplicity has been a focal point of the analysis. Relative to the work of Bianchi ([12]), we extend his results in a number of directions. First, we provide a complete qualitative theory of how the *set* of RE vary in the deep parameters of the model, provide a new successive approximation/generalized "time-iteration" algorithm for *computing* least and greatest RE as well as their equilibrium comparative statics. Second, as in Bianchi ([12]), we study global stochastic equilibrium properties that are induced by the *set* of RE within the context of a GME representation. The GME setting allows us to address many new and interesting questions. For example, we can compare the stochastic properties of stationary equilibrium and it's relationship with the long-memory structure into the GME representation of SCE, and then compare the stationary equilibrium properties of such GME representations to the properties of stationary Markov equilibria induced by the set of RE of the model. We find that incorporating long-memory is important in fitting the anatomy of the observed financial crises.

Relative to the work of Schmitt-Grohé and Uribe (e. g., [70], [71])), the approach taken in this pa-per is very different, but in a very specific sense complementary. First, exploiting the same source of equilibrium multiplicities they discuss in the context of SCE, we globalize their results of existence of multiple equilibria in these models. Second, we take a very different quantitative approach to studying the properties of stochastic dynamic equilibria by using GME methods. In Schmitt-Grohé and Uribe ([70], [71]), their approach to characterizing the existence of multiple equilibrium is built upon the *deterministic* versions of the model, and in particular sequential equilibrium behavior "local" near a (deterministic) steady-state.[6] Our results are never "local" or "deterministic", rather they global and stochastic, and we show the sources of multiple equilibrium stem from the existence of an implicit *equi-librium complementarity* in prices generated between equilibrium collateral constraints in the *stochastic* version of the model and household consumption/debt choices. Our arguments prove the existence of *multiple stochastic stationary equilibria*. Our approach to characterizing dynamic stochastic equilibrium therefore makes no mention of local steady state structure, local sunspot equilibria, or even the determin-istic steady-state, builds a theory of stationary stochastic equilibrium from *arbitrary initial conditions,* provides a systematic theory of modeling multiple equilibrium, memory and the associated stationary equilibrium from vantage point of selectors from the set of GME representations of SCE. What is critical in our approach to study stochastic properties of GME selections using the "hit" times for equilibrium collateral constraint relative to stochastic equilibria, and use the borrowing constraint itself to regenerate the equilibrium Markov process associated with the set of GME.

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[4]See also the recent paper of Pierri, Montes, Rojas, and Mira ([59]) for further discussion of the nature of Sudden Stops, and and discussion of the related empirical literature.

[5]For a discussion of this extensive recent theoretical literature, see Bianchi and Mendoza ([16]) and Schmitt-Grohe and Uribe ([70]).

[6]These papers then characterize stochastic SCE dynamics using technique building "stationary sunspot" approaches (e. g., see related work in Woodford ([78]) and Schmitt-Grohé ([66]), for example).

It bears mentioning, our GME approach is general and can be potentially be applied in other models of Sudden Stops found in the existing literature. The GME approach in this paper can be extended to some Sudden Stop models with elastic labor supply and production. Versions of the models with production include the early papers of Mendoza ([46]) and Mendoza and Smith ([48])), as well as the series of interesting papers by Benigno, Chen, Otrok, Rebucci, and Young (e. g., see [8], [9], [10]). and Deveraux and Yu ([26]).[7]

Our paper also is related to an emerging literature that seeks to characterize dynamic model with equilibrium borrowing constraints (and/or occasionally-binding constraints). Relative to this literature, we provide a new set of methodological tools for characterizing the RE and SCE in models with equilibrium price-dependent collateral constraints. Our multistep fixed point approach to RE can be show to be useful on other related dynamic equilibrium models where stationary equilibrium partitions into states where collateral constraints are "slack" versus, "binding", and include models of credit cycles in the spirit of Kiyotaki and Moore ([36]),[8] models of financial frictions and production with collateral constraints such as Moll ([54]), or models of self-fulling credit cycles such as Azariadis, Kaas, and Wen ([7]).

This paper also contributes to the growing literature on GME methods in the study of self-generation via strategic dynamic programming approaches to dynamic stochastic general equilibrium models (DSGEs) in the literature. These self generation techniques were first introduced in the repeated games in the Abreu, Pearce, and Stacchetti ([1], [2]) and are related to implementations of the methods for studying the existence of Markovian equilibrium in dynamic stochastic models/games found in Blume ([17]) and Duffie, Geanokopolis, Mas-Colell, and McLennan ([27]). Approaches to making operational these meth-ods in the context of GME representations and enlarge state spaces stems from the work of Kydland and Prescott ([40]), Phelan and Stacchetti ([57]), Kubler and Schmedders ([39]), Feng, Miao, Peralta-Alva, and Santos ([29]), and Cao ([19]). A novel aspect of this paper is that by exploiting the structure of equilibrium price-dependent collateral constraints, we are able to propose a systematic approach to SCE selections, based upon ergodic GME selectors.

At least since the paper of Blume ([17]), when studying questions of equilibrium stochastic stabil-ity, it has along been recognized the trade off between the multiplicity of sequential equilibria and the continuity properties of the associated recursive representation. In other words, the presence of multi-ple equilibrium can generate discontinuities if dynamic equilibria over minimal state space equilibrium transition functions. In some cases, by enlarging the state space, it is possible to obtain a continuous Markov equilibrium (see Martinez and Pierri, [31]); but unfortunately, there is no general theory about how to do this (see Kubler and Schmedders, ([38]), for a counter-example). Interestingly, in the context of models with price dependent collateral constraints, we show that the GME methods we develop in section 4 are critical in the presence of this type of discontinuities, which may affect the existence of an ergodic equilibrium. Section 5 of the paper then shows that it is possible to obtain multiple equilibrium when the collateral constraint binds for a *wide range of parameter values.* We show that the number of possible exogenous shocks is critical to generate multiplicity. Our approach to this issue therefore is novel in a *stochastic* setting (for instance, Schmitt-Grohé and Uribe ([70]) discussed these issues in the setting of a non-stochastic steady state of the economy).

Finally, our work also directly related to the literature on the equilibrium comparative statics in dynamic economies using monotone-map methods (or "time-iteration") methods.[9] In a recent paper, Datta, Reffett, and Wozny ([24]) propose a multistep monotone-map/time iteration method that proves especially suited for dynamic models with multiple RE. Our paper extends the class of multi-step monotone-map methods to dynamic models with equilibrium price-dependent collateral constraints. In addition, our paper is related to Mirman, Morand, and Reffett ([53]), Acemoglu and Jensen ([3]), and Datta, Ref-fett, and Wozny ([24]) as it provides sufficient conditions for monotone dynamic equilibrium comparative statics in the deep parameters of the economy. In particular, this paper extending these results into the dynamic models with equilibrium price-dependent borrowing constraints.

[7] See Pierri and Reffett ([60]) for a discussion.

[8] See the survey of Gertler and Kiyotaki [33] for a nice discussion of this large literature, as well as Kiyotaki and Moore ([37]) for recent work along these lines.

9 This literature start with the papers of Coleman ([20], [21]), and has been extended in Reffett ([61]), Datta, Mirman, and Reffett ([23]), Morand and Reffett ([55]), and Mirman, Morand, and Reffett ([53]), among many others.

The paper is organized as follows: section 2 discusses the model, section 3 presents the results for the RE, section 4 contains the results for the GME and section 5 contains applications to a) different anatomies of short run crises, b) multiple equilibrium in a stochastic setting, c) solve the model, compute and simulate ergodic, stationary and non-stationary equilibrium. Section 6 concludes.

2. The Model and Sequential Competitive Equilibrium

We consider the endowment version of the two-sector small open economy model studied in Bianchi ([12]).[10] This is a workhorse model in the international finance and macroeconomics literature for the study of emerging market debt crisis and sudden stops and abstracts from production.[11] We choose the endowment version this model as it allows us to focus on the issues at hand in this paper, the importance of modeling of memory in the presence of multiple equilibrium, both from a qualitative and quantitative perspective. As is well known, what is critical in these models is presence of occasionally binding price-dependent *equilibrium* collateral constraints, as these constraints introduce pecuniary externalities in equilibrium structure of tradeable consumption via the relative price of nontradeable endowments.

The model is a small open economy with a fixed interest rate. Time is discrete over an infinite horizon and indexed by $t \in \{0, 1, 2, ...\}$. There is a representative agent and two sectors of perishable goods in the economy, a tradable consumption good y_t^T and a non-tradable consumption good y_t^N. Each household is endowed a strictly positive amount of each good in each period. Upon receiving their current period endowments, households sell endowments at current market prices and choose consumption of both goods where the consumption of tradeable and non-tradeable is denoted, respectively, by c_t^T and c_t^N. It turns out to be useful to take as the numeraire the tradable good, so the relative price of non-tradeable relative to the numeraire tradeable in period t is denoted by p_t.

Household preferences are defined over infinite sequences of dated consumption vectors of tradeable and non-tradeable goods $c_t = (c_t^T, c_t^N) \in X \supset \mathbf{R}_+^2$ where X is the commodity space for consumption of tradeable and non-tradeable in each period, and are assumed to be time separable with subjective discount factor $\beta \in (0, 1)$. These preferences are represented by a nested utility function which is a composition of two functions: a utility over composite consumption $U : \mathbf{R} \longrightarrow \mathbf{R}$, and an aggregator $A : X \longrightarrow \mathbf{R}$ over tradeable and non-tradeable consumption $c_t = (c_t^T, c_t^N)$, where the preferences $U(A(c))$ gives the instantaneous utility of the vector of consumption $c \in X$ in any period. Then, lifetime discounted expected utility preferences of a typical household are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{1}$$

where the mathematical expectation operator here is taken over the stochastic structure of uncertainty with respect to the date 0 information.[12]

Uncertainty in the economy is modeled as an iid stochastic process governing tradeable endowments $y = \{y_t^T\}_t$ where each element of sequence y has distribution given by the measure $\chi(\cdot)$. Here, for convenience, we assume the sequence of non-tradeable endowments $\{y_t^N\}_t$ is non-stochastic as it plays no role in the characterization of stochastic equilibrium dynamics in this paper. We assume further that the realizations for tradeable endowments in any period denoted by $y_t^T \in Y$ where the shock space Y

[10] Studies of sudden stops using the endowment version of the two-sector model of Bianchi ([12]) are numerous. See Bianchi, Liu, and Mendoza ([14]), Schmitt-Grohe and Uribe ([69], [70], [71]), Arce, Bengui, and Bianchi ([6]), and Lutz and Zessner-Spitzenberg ([43]), among many others.

[11] The model (and our approach) can be extended to models with production such as Mendoza ([46]), Mendoza and Smith [48]), Biachi and Mendoza ([13]), the series of papers by Benigno, Chen, Otrok, Rebucci, and Young (e. g., [8], [10]), and Devereux and Yu ([26]), among many others.

[12] A typical functional form for the consumption aggregator $A(c)$ in the literature is the Armington/CES aggregator

$$c_t = A(c_t^T, c_t^N) = [a c_t^{T\ 1-\frac{1}{\xi}} + (1-a) c_t^{N\ 1-\frac{1}{\xi}}]^{\frac{1}{1-\frac{1}{\xi}}}$$

with $\xi > 0$, $a \in (0, 1)$, which is increasing, strictly concave, and supermodular on X when $X = \mathbf{R}_+^2$ with its product order.

is finite set.[13] By an application of standard results, these assumptions on shocks imply it is possible to define a stochastic process $(Y_\infty, \Omega, \mu_{y_0^T})$ which takes realizations in each period in Y. [14] Given this fact, denote by Y_∞ the space of infinite sequences in Y, and assume $y_0^T \in Y$ is the initial condition for this stochastic process for tradeable.

Household face a sequence of budgets constraints when making their sequential choices for consumption and debt over their lifetimes. In particular, given a candidate price sequence $p = \{p_t\}_{t=0}^{\infty}$, and denoting the net debt position for a typical household with debt borrowed at date t but maturing at date $t+1$ by d_{t+1}, the budget constraint for a household in any period t is given by:

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R} \tag{2}$$

Where agents are allowed to borrow or lend at a fixed interest rate $R = 1 + r$ and, as this is a small open economy, R is taken as given. The sequential budget constraints here follow the timing convention used in Schmitt-Grohé and Uribe ([70], [71]), and assume consumption and income decisions are taken at the beginning of the period, and interest is then paid/earned over that same period. We adopt this timing only because it proves to be convenient in characterizing the structure of dynamic equilibrium.[15] In addition to the budget constraint in (2), a typical household agent also faces a period by period flow collateral constraint on debt given by:

$$d_{t+1} \leq \kappa(y_t^T + p_t y_t^N) \tag{3}$$

where $\kappa > 0$ [16]

A few remarks on this model are in order at this stage. First, the collateral constraint in (3) is an occasionally binding price-dependent collateral constraint. That is, in what states the collateral constraint binds is *endogenous* and an *equilibrium object.* This fact, in conjunction with it's implicit equilibrium complementarities the constraint will be show to induce will be the key friction in this class of economies driving the existence of multiple SCE. That is, as has been shown in for example Bianchi ([12]) and Schmitt-Grohé and Uribe ([69], [70],[71]), the presence of this equilibrium price-dependent collateral constraint introduces a *pecuniary externality* into this dynamic equilibrium structure of economy. What is new in the present paper is that we shall show this pecuniary externality induces an equilibrium pecuniary complementarity, which is precisely an equilibrium single-crossing condition between individual tradeable consumption and their per-capita aggregate counterparts in the equilibrium collateral constraints in states when the collateral constraint bind. This equilibrium single-crossing condition appears via the following simple mechanism: as the relative price of tradeable to non-tradeable p_t is increasing in the per-capita level of tradeable consumption in equilibrium, when agents believe in equilibrium paths for per-capita tradeable consumption will be "higher", the equilibrium collateral constraint is relaxed, allowing household tradeable consumption to expand via access to additional debt and tradeable consumption. This dynamic equilibrium complementarity generates an implicit (equilibrium) monotone relationship between per-capita tradeable and household tradeable consumption, and becomes the single *global* source of multiple dynamic equilibria in this economy in both sequential competitive equilibria, as well as (minimal state space) recursive equilibrium. For example, in a RE, we shall show that the equilibrium collateral constraints are *ordered.*

Second, in many alternative versions of these models, agents have access to at least two assets, with only one of them subject to price-dependent equilibrium collateral constraint (e. g., see Mendoza and Smith ([48])). This choice of multiple assets is relevant often to understand the quantitative implications

[13] All the results of sections 2 and 3 on the existence of sequential competitive equilibrium and recursive equilibrium hold for more general shocks (i.e., endowment processes for tradeables consumption that follow a first order Markov process with stationary transition $\chi(y^T, y^{T\prime})$), but with substantive difference in proof. See Pierri and Reffett ([60]).

[14] e. g., see Stokey, Lucas and Prescott ([74], chapter 7)).

[15] Bianchi ([12]) uses a slightly different timing convention, but it turns out this timing convention is without loss of generality in our case (see, for example, Adda and Cooper ([4]) for a detailed discussion of this matter).

[16] As is well-known in this literature, one can write down more fundamental versions of this model where this debt constraint emerges as an equilibrium object from the primitives of the underlying economy. For our purposes, we just follow the literature and impose this form of a price-dependent collateral constraint.

of borrowing constraints when such borrowing constraints bind in equilibrium, allowing one to further capture the "spiraling" process of "fisherian deflation" associated with a sequence of descending equilibrium prices. In this case, the deflation is caused by a real exchange rate depreciation occurring after a balance of payment crises. In models with two assets, this "deflation" simply occurs as a consequence of an equilibrium sell-off in the pledgeable assets in order to obtain resources needed to allow agents to smooth consumption. Further, the theoretical approaches to constructing and characterizing dynamic equilibrium in such multi-asset models does not change substantially, so in this paper we focus on the single asset case which facilitates comparisons with the existing literature.

2.1 Existence of Sequential Competitive Equilibrium

We first consider sufficient conditions for the existence of a sequential competitive equilibrium (SCE). In a SCE, the representative household takes as parametric an interest rate R, a level of initial debt $d_0 \in D$ (where $D \subset \mathbf{R}$ is a compact set of debt states which will be constructed in a moment), a stochastic process governing $y^T = \{y_t^T\}_{t=0}^{\infty}$ conditional on an initial level of tradeable endowment $y_0^T \in Y$ with $y_0^T > 0$, a constant level of non-tradeable endowment $y_t^N = y^N$, and a history-dependent sequence of measurable prices $p = \{p_t(y^t)\}_{t=0}^{\infty}$, where we use the notation $y^t = \{y_0^T, y_1^T, .., y_t^T\}$ to denote the history of tradeable endowment realizations up to period t, and it maximizes lifetime utility in (1) subject to constraints (2) and (3) for each time period. That is, given $R \geq 0$, d_0, the stochastic processes p and y, the household solves the following:

$$V^*(s_0, p, R) = \max E_0 \sum_{t=0}^{\infty} \beta^t U(A(c_t)) \tag{4}$$

$$c_t^T + p_t c_t^N + d_t = y_t^T + p_t y_t^N + \frac{d_{t+1}}{R}; \quad d_{t+1} \leq \kappa(y_t^T + p_t y_t^N),\ t \in \{0, 1, 2, ...\} \tag{5}$$

where the initial states are $s_0 = (d_0, y_0^T)$, and $y_0^T \in Y$. We denote the optimal policy sequences for consumption and debt achieving the maximum on (4) by

$$c^*(s_0, p, R) = \{c_t^*(s_0, p, R)\}_{t=0}^{\infty};\ d^*(s_0, p, R) = \{d_t^*(s_0, p, R)\}_{t=0}^{\infty}\} \tag{6}$$

where in a moment we shall impose sufficient convexity and continuity conditions on the primitive data of the model such that (a) the value function $V^*(s_0, p, R)$ is finite, and (b) optimal sequences $c^*(s_0, p, R)$ and $d^*(s_0, p, R)$ are well-defined and unique.

Further, under standard convexity, continuity conditions, and continuous differentiability assumptions on preferences, by well-known arguments (e. g., Rincon-Zapatero and Santos ([62]), theorem 3.1) (e. g., see Assumptions 1 and 2 below), one can show there exists a well-defined standard Lagrangian formulation for the sequential primal problem in (4) with (summable) dual variables $\beta^t \lambda_t$ and $\beta^t \lambda_t \mu_t$ associated with the sequence of constraints (2) and (4), respectively. Noting our constrained system satisfies sequential linear independence constraint qualifications, strong duality holds between the resulting Lagrangian formulation and the primal program in (4) and the infinite dimensional system of KKT multipliers are well-defined and unique. We can then formulate the system of first order conditions for this problem in a sequential competitive equilibrium using the Lagrangian dual as follows: the optimal stochastic processes $c^*(s_0, p, R)$ and $d^*(s_0, p, R)$ satisfy

$$\lambda_t^* = U'(A(c_t^*))A_1(c_t^*) \tag{7}$$

$$p_t = \frac{A_2(c_t^*)}{A_1(c_t^*)} \tag{8}$$

$$[\frac{1}{R} - \mu_t^*]\lambda_t^* = \beta E_t \lambda_{t+1}^* \tag{9}$$

$$\mu_t^*[d_{t+1}^* - \kappa(y_t^T + p_t y_t^N)] = 0, \mu_t^* \geq 0 \tag{10}$$

A sequential competitive equilibrium for this economy is then defined as follows: [17]

[17] In the appendix, where we prove existence of sequential competitive equilibrium, we had the formalities of measurability of SCE processes more rigorous. See the details there.

Definition 1 *Sequential Competitive equilibrium (SCE) is a collection of progressively Ω-measurable random variables for consumption* $c^*(s_0, p^*(s_0, R), R)$*, debt* $d^*(s_0, p^*(s_0, R), R)$*, and relative prices of non-tradeable to tradeable consumption* $p^*(s_0, R)$ *such that: 1) the representative agent chooses* $c^*(s_0, p^*, R)$*,* $d^*(s_0, p^*, R)$ *to solve (4) given* s_0 *at* $p^*(s_0, R)$ *such that* $V^*(s_0, p, R)$ *is finite and equations (7-10) hold, and 2) markets clear* $c_t^{N*}(s^t) = y^N$ *where* y^t *holds* $\mu_{y_0^T}$*-a.e*

We first consider the question of existence of SCE. Before mentioning the proposed sufficient conditions, we need to mention a few key technical issues that arise when studying the structure of SCE in stochastic infinite horizon debt models. A first critical object to place enough structure on the model we can characterize the (nonempty) set of *stochastic steady state* distributions (i.e., the set of *stationary equilibrium).* This stationary equilibrium can be associated with either the SCE and/or the recursive equilibrium of the model. As a stationary equilibrium is *global stochastic* equilibrium object, we need to impose sufficient structure on the model such that we can obtain sufficient boundedness (actually, compactness) for the models stochastic equilibrium dynamics.[18] In the existing literature, the question of how to provide lower and upper bounds for stochastic equilibrium paths for endogenous variables in small open economy frameworks such as ours has not been addressed. Yet, it is well known that in stochastic models of debt, issues related to lower and upper bounds can be delicate.

To impose such sufficient structure on the economic primitives to obtain such lower and upper bounds on stochastic sequential equilibrium dynamics, in the extensive existing literature on stochastic debt models, typically sufficient conditions involve bounds on the marginal utility of c^T in addition an otherwise standard conditions on these model in applications. These bounds are guaranteed to hold by slightly modifying standard preferences in the literature in a manner we will define in a moment when we state our sufficient conditions for existence of SCE. What is critical to note here is that these type of restrictions *do not affect the stochastic dynamic behavior* of the model when compared to the more "traditional" CES preferences (as we shall discuss in the quantitative section of the paper in section 5). They are simply critical to insure the necessary compactness of the sequential equilibrium set, as well as provide the basis for the existence of a stationary and compact state space in both the SCE and recursive formulations of the dynamic equilibrium of the model. That is, when studying the quantitative stochastic properties of these models in calibrated settings, compactness of the state space if often present without these types of "boundary" conditions on preferences.

The rest of the section is devoted to show that any sequential competitive equilibrium in this economy is compact in the sense that realizations of equilibrium random variables are all contained in a compact subset of a finite dimensional space. This property is essential to define a proper Markovian representation. We the state our assumptions on the primitive data of the model as follows:

Assumption 1:
The functions $U(x)$ *and* $x = A(c)$ *satisfy the following: a.i)* $lim_{x \longrightarrow \infty} U'(x) = 0$ or *a.ii)* $\exists x \in X$ *such that* $\forall y \in B_\epsilon(x) \quad U(y) \leq U(x) \quad \epsilon > 0$*, a.iii) Let* $A(.,.)$ *be a function mapping* $X \mapsto \mathbf{R}_+$*, where* X *is the consumption space,* $A(., c^{NT}) \neq 0$ *for any* $c^{NT} > 0$ *b)* $\mathbf{R}_+^2 \subset X, X$ *open, c)* $A_1(c) = A_1(c_1)$[19]*, d)* $A_2(c) = A_2(c_2)$*, e)* $lim_{c_1 \longrightarrow \infty} A_1(c_1) = cl_1 > 0$*, f)* $lim_{c_1 \longrightarrow 0} A_1(c_1) = cu_1 < \infty$ *, g)* $U(A(c)) : X \longrightarrow \Re$ *is* C^1*, continuous, strictly increasing, concave.*

When we state the existence of recursive equilibria in the next section, as the equilibrium dynamical system are greatly simplified, we can also allow for the following case of preferences:

Assumption 2.
$U(A(c^T, c^{NT}))$ *satisfies the Inada condition for tradeables consumption: for* $c = (c^T, c^{NT})$*,* $c^{NT} > 0$*,* $\lim_{c^T \to 0} U'(A(c))A_1(c) \to \infty$.

[18] Questions concerning the existence and characterization of the stochastic steady state (e.g, stationary equilibrium) the results in Schmitt-Grohé and Uribe ([67]) involve the *non-stochastic steady state*, which is not relevant for the purpose of this paper, or the formulation of Assumption 1. We would like to thank Javier Garcia Cicco mentioning this fact to us.

[19] The notation $A_{\ j}$ denotes the partial derivative with respect to the $j = 1, 2$ argument of A.

We make a few remarks on these assumptions. Assumption 1.a.ii can be used to prove the existence of any SCE only in the case where $\beta R \geq 1$, which is typically missing in the literature. It simply says that lifetime preferences must have an asymptotic "satiation point" $x \in X$. As these preference structure affects the underlying stochastic process significantly, attracting equilibrium paths to the satiation point, we defer the treatment of this case to a companion paper. Thus, we focus in the traditional case (i.e. $\beta R < 1$). Assumption 1.a.iii gives us some flexibility in the choice of U, as it automatically implies that $U'(c)$will be well behaved when $c \longrightarrow 0$. Assumption 1.b, 1.e, and 1.f insures sequential equilibrium prices are bounded above and bounded below away from zero. This, in turn, will imply that in a SCE, debt is bounded above due to the collateral constraint in (3) and bounded below as wealth will be finite at every possible node. It is possible to relax Assumption 1.e and 1.f by restricting the superdifferential, the set which includes all possible supergradients, of the concave mapping c on X to be compact. However, in applications, preferences are assumed to be stronger than Assumption 1.g and continuously differentiable. In this case, the strengthening of Assumption 1.e and 1.f can be done without loss of generality.

We also remark, when studying the existence of RE, we can replace Assumption 1.f with Assumption 2 (the standard Inada condition case). This assumption imposes, in effect, and Inada condition on $A(c)$ in tradeables consumption. We can relax Assumption 1.f when constructing RE as the characterization of stochastic equilibrium dynamics can be greatly sharpened (in particular, its uniform interiority of tradeable consumption) allowing us to characterize the (strong) interiority properties of stochastic dynamic equilibrium consumption paths. Finally, Assumption 1.g is standard in the general equilibrium literature (see for instance Braido ([18])) . In applications, Assumption 1.g will be insured by assuming that the set X is uniformly bounded below, $U : \mathbf{R} \longrightarrow \mathbf{R}$and $A : X \longrightarrow \mathbf{R}$ are both increasing, concave and twice-continuously differentiable.

We now state our first critical result in building SCE: [20]

Lemma 2 *Suppose $\beta R < 1$. Under assumptions 1-a.i, 1-a.iii, 1-b to 1-g, **if** a SCE exists, it must satisfy* (54),(55) *and $lim_{t \longrightarrow \infty}\ \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$. Moreover, **if** (c, d, p) in is a SCE, then $[c(y^t), d(y^t), p(y^t)] \in K_1 \times K_2 \times K_3 \subset \Re^4 \quad y^t-$a.e. and uniformly in $[y_0, d_0] \in Y \times K_2$, where $K_1 \times K_2 \times K_3$ is compact.*

We mention that following the results in [62], Lemma 2 gives an equivalent characterization of the SCE in terms of the equations (7)-(10) with well defined sequential KKT multipliers that we use in the appendix. Also, notice that this lemma does not guarantee the existence of the SCE, which will be considered in Theorem 3 below.

As we are allowing the cardinality of Y to be arbitrarily large, proving the existence of this type of equilibria can be rather challenging (See, for instance, Mas-Collel and Zame, [44]). Once the almost every where compactness of any "suitable candidate for equilibria" (c, d, p) is proved using lemma 2, the existence of the SCE will be proved assuming that Y is a finite set. MasColell and Zame ([**?**]) needed to state by assumption the uniform compactness of the SCE for the case of uncountable shocks. As it is natural, to show the existence of a recursive representation in this last case, we must impose the same assumption. The sequence of remarks which follows the Theorem 3 go deeper into this topic.

Finally, Lemma 2 shows necessity of Assumption 1 for the existence of a sequential equilibrium. The following theorem shows sufficiency.

Theorem 3 *Suppose $\beta R < 1$ and the Y is a finite set. Under Assumptions 1-a.i,1-a.iii, 1-b -1-g, there exist a SCE*

The following set of remarks are connected with the results shown above.

[20]All the proofs of the Lemmas and Theorems in the paper are in the Appendix.

Remark 4 *The existence of the SCE for an uncountable set of shocks could be shown using the arguments in Mas-Colell and Zame ([44]). The authors showed existence assuming the uniform compactness of the equilibrium variables for an economy with an uncountable number of i.i.d shocks and finite time. It remains to be verified if the uniform compactness assumption, needed because of the almost everywhere nature of the characterization of the SCE even in finite time, is sufficient to extend the proof to an infinite horizon. However, this paper shows that, in order to study the long run properties of the model, it suffices to assume a finite set of shocks.*

Remark 5 *Once the dual minimal state space recursive representation of equilibria is shown to exist in section 3, it is possible to apply standard envelope arguments. Equipped with these results and assuming the absolute continuity of the Markov operator χ, it is immediate to show the existence of a subset of all possible SCE in Definition 1 with Y uncountable and compact and Markov exogenous shocks. Because of lemma 2, the equilibrium will be compact almost everywhere. This result constitutes an extension to Mas-Collel and Zame (1996) and to our knowledge, it has not been shown before.*

2.2 Discussion

The previous subsection characterizes the sequential competitive equilibrium and stationary equilibrium (e. g., stochastic steady state) for $\beta R < 1$ with Y finite. These findings constitute an extension with respect to previous theoretical results in the literature (e. g., approaches appealing to stationary sunspot equilibria as Schmitt-Grohé and Uribe ([67])), which provide conditions to insure a stationary long run behavior around a non-stochastic steady state via sunspots). Taking into account that most applied papers evaluate the effects of alternative economic policies using simulations generated from a computed model, the existence of a well behaved (i.e. stochastic) steady state is critical as it allows for the possibility of convergent simulations from *arbitrary initial conditions.* .

In this paper, we do not discuss existence issues for the case that $\beta R \geq 1$. In a related paper, Pierri and Reffett ([60])) identify sufficient conditions for extending the results to this case (including setting with more general shocks and continuous shock spaces). The critical complication for introducing $\beta R \geq 1$ is obtaining compactness when studying the long-run behavior of such a model relative to stationary equilibrium. Turns out, if we endow the model with a satiation point, the equilibrium has a degenerate steady state as consumption converges to a Dirac measure a.e. It is possible, as in Hansen and Sargent ([35]), to allow for a generalization of this last type of equilibrium by assuming that the satiation point (called "bliss point") is a random variable. Although it can be useful in some applications (i.e. asset pricing with no trading, etc.), this type of equilibrium has really restrictive dynamics. Taking into account the question at hand, we defer the discussion of this case to a separate paper.

An example of the restriction on preferences implied by Assumption 1 can be seen in the table below.

Pref.	Does SCE Exist?	$c > 0$?	MU bounded?	Are Homothetic?
CD	Unknown	YES	NO	YES
LOG	Unknown	YES	NO	YES
CES	Unknown	YES	NO	YES
Mod. CD	Theorem 1	YES	YES, above	NO
Mod. LOG	Theorem 1	YES	YES, above	NO
Mod. CES	Theorem 1	YES	YES, above	NO
Mod. CES 2	Theorem 1	YES	YES, AF zero	NO

Table 1: Restriction on preferences

where the abbreviation "MU" stands for "marginal utility" and "AF" for "away from" . In the appendix, we provide a concrete parametrization of preferences for each of the cases presented in table 1.

Note that Theorem 3 requires MU to be bounded above and away from zero. The requirement that MU is bounded above breaks the homotheticity of preferences (i.e., Inada conditions), but the assumption of bounded away from zero allows for the homothetic case. As we will later show, the existence of minimal state space recursive equilibrium can be shown even without an finite upper bound on marginal utility; but the existence of a maximal consumption still requires MU of consumption be bounded away from zero.

More to the point, the proposed utility functions can be made arbitrarily close to their homothetic counterpart and the numerical section in this paper, which suggests that assuming boundedness, instead of imposing restrictions on marginal utility, and using standard CES preferences works in practice. We need bounds on marginal utility to ensure the compactness of the equilibrium set in order to complete the existence proof of the SCE. As the numerical results in this paper use extensively the quantitative implications of this last type of equilibrium, our simulations in this paper in section 5 are well behaved under the standard CES preferences.

Finally, the non-homotheticity of preferences is imposed to obtain boundedness (hence, compactness). In the literature preferences are typically homothetic. However, Assumption 1 gives an additional source of nonlinearity to the equilibrium dynamics as income effects are not restricted by this hypothesis. Straub ([73]) found that under homothetic preferences consumption is linear in permanent income, a fact which is at odds with data. He extends the canonical precautionary savings model with heterogeneous agents to include non-homothetic preferences. To our knowledge, the literature on macroeconomic crises has not explored the implications of non-homothetic preferences on the empirical performance of standard models.

3. Existence and Characterization of Recursive Equilibrium

Next we study the existence (minimal state space) RE for this class of models. To construct RE, we begin by representing the aggregate economy recursively on a minimal state space. The minimal state space for the (set) of RE will consist of current state variables summarizing the individual state of a representative household and the aggregate state of the aggregate economy. In this economy, in any period of a RE, a household enters the period with an individual level of debt $d \in \mathbf{D} \subset \mathbf{R}$, where $\mathbf{D}$ is compact [21], as well as an endowment of tradeable and non-tradeable denoted by the vector $y = (y^T, y^N)$, where $y \in \mathbf{Y} \times \{\mathbf{y}^{NT}\} \subset \mathbf{R}^2_{++}$. So the *individual state* of the household is characterized by the vector $(d, y) \in \mathbf{D} \times \mathbf{Y}$. At the beginning of any period, a typical household also faces an aggregate economy in an *aggregate state* consisting of per-capita aggregate measures of each of these individual state variables That is, the aggregate state variable is a vector $S = (D, Y) \in \mathbf{D} \times \mathbf{Y} = \mathbf{S}$, where $D \in \mathbf{D}$ is the per-capita level of aggregate debt, Y^T (resp, Y^N) are the per-capita endowment draws for tradeable (resp., non-tradeable) endowments with vector $Y = (y^T, y^{NT}) \in \mathbf{Y} \times \{\mathbf{y}^{NT}\} \subset \mathbf{R}^2_{++}$. Therefore, the state of a household entering any given period in a RE will be denoted by $s = (d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$.

We now construct a recursive representation of the aggregate economy on this aggregate state space $S = (D, Y) \in \mathbf{S}$. Anticipating the structure of RE, first notice the relative price for non-tradeable to tradeable (denoted by $p(C^T)$) in any RE will be equal to equilibrium marginal rate of substitution between non-tradeable and tradeable: i.e,

$$\frac{U_2(C^T, y^N)}{U_1(C^T, y^N)} = \frac{A_2(C^T)}{A_1(y^N)} = p(C^T) \tag{11}$$

Where C^T is some per-capita aggregate level of tradeable consumption, and we impose in any RE the fact that $c^N = y^N = Y^N$, where Y^T is constant (and we suppress the dependence of p on the assumed constant level of Y^N). Under the supermodularity and concavity conditions in Assumption 1 and 2 relative to the composition $U(A(c))$, we have $p(C^T)$ is increasing in C^T.[22] Using equation (11), we can generate a recursive representation of sequential prices $p = \{p\{S_t\}\}$ by constructing candidate laws of

[21] Under Assumption 1(a)-(e) and 1(g), we can take $\mathbf{D}$ compact given results on existence of SCE in Lemma 1 and Theorem 1.

[22] For example, much of the applied literature uses Armington aggregators for the mapping $A(c)$, so the relative price p

motion for the per-capita aggregate debt D, which used in conjunction with realizations of endowment $\{Y_t\}$ generate realizations of sequential paths for prices $p = \{p(S_t)\}_{r=}^{\infty}$. To do this, define a collection of candidate socially feasible per-capita tradeable consumption $C^T : \mathbf{S} \rightarrow [0,c^{\max}] \subset \mathbf{R}_+$ denoted by $\mathbf{C}^f(\mathbf{S})$:

$$C^T \in \mathbf{C}^f(\mathbf{S}) = \{C^T(S)|0 \leq C^T(S) \leq c^{\max},\ C^T \text{ is increasing in } Y, \text{ decreasing in } D \text{ and jointly continuous}\} \tag{12}$$

where from the previous section, under Assumption 1, $c^{\max}$ is finite.[23] Endow the space $\mathbf{C}^f$ with the standard pointwise partial order $\geq$. Then, the space $(\mathbf{C}^f, \geq)$ is a nonempty lattice in this partial ordering.[24] For any element $C^T \in \mathbf{C}^f$, when $C^T(S) > 0$, we can identify the implied law of motion for per-capita debt D in a RE by using equilibrium versions of the household's budget constraints and collateral constraints: i.e., the per-capital debt evolves according to:

$$D' = \Phi(S; C^T) = \inf[R\{C^T(S) - Y + D\}, \kappa\{y^T + p(C^T(S))y^N\}],\ C^T \in \mathbf{C}^f \tag{13}$$

where R is the current interest rate. As D is the only endogenous aggregate state in this economy, in conjunction with the primitives of the stochastic process on the endowment shocks Y, we now have a full characterization of the stochastic transition structure of the aggregate economy in any candidate RE $C^T(S) \in \mathbf{C}^f(\mathbf{S})$. We use this representation of the aggregate economy to parameterize a recursive representation of household dynamic program in a moment.

Before continuing, we make two remarks concerning the function space $\mathbf{C}^f$ used at this stage to parameterize the aggregate economy in a candidate RE. First, we must study the existence of RE within a *strict (closed) subset* of functions $\mathbf{C}^*(S) \subset \mathbf{C}^f(\mathbf{S})$, (where the construction of this subspace $\mathbf{C}^*(S)$ will be discussed in great detail momentarily). There are many reasons for this fact.

First, there exist elements $C^T \in \mathbf{C}^f$ which do not admit admissible SCE prices $\{p(C^T(S_t)\}$. In particular, for $C^T(S) \in \mathbf{C}^f$, we only impose $0 \leq C^T(S) \leq c^{\max} < \infty$, and as $p(0)$=0 for many parameterizations of preferences, $p(C^T) = 0$ whenever $C^T(S) = 0$ in any state. This obviously not consistent with households facing a compact sequential budget constraint (or a compact budget correspondence for any RE formalization). Therefore, it must be for an RE price system p satisfy the basic conditions needed for any SCE, any RE must have $C^{T*}(S) > 0$ for any state S, so $p^u \geq p^* \geq p^l > 0$ in all states $S \in \mathbf{S}$.[25]

Second, the space $\mathbf{C}^f$ does also not capture all of the *necessary* structural restrictions of *any* RE policy functions for tradeable goods. For example, if we have *multiple* RE, RE will in general be *discontinuous;* but, as will be clear in a moment, in any RE, the RE *cannot* be discontinuous over *individual states.*[26] So, in any RE policy for tradeable consumption $C^{T*} \in \mathbf{C}^* \subset \mathbf{C}^f$ given as a fixed point of the mapping $C^{T*}(S) = c^{T*}(d, y, S; C^{T*}(S))$ where when $d = D$, and $y = Y$ and $c^{T*}(d, y, S; C^T)$ is the typical household's policy function, *all* the discontinuities over the RE state space must be isolated to *only aggregate states S*. This is a somewhat subtle issue, and often not handled explicitly in the literature when constructing RE. It should also be clear, this requirement can be *very demanding* in settings with multiple equilibria.

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is just:

$$p(C) = \frac{1-a}{a}\left(\frac{C^T}{Y^N}\right)^{1/\xi}$$

Under Assumption 1.g, for constants α^T>0 and $\alpha^N > 0$, we could have utility such that the price function is:

$$p(C) = \frac{1-a}{a}\left(\frac{C^T + \alpha^T}{Y^N + \alpha^N}\right)^{1/\xi}$$

such that marginal utility (and hence, prices) are bounded for all $C^T \geq 0$. Note, this special case is is important to keep in mind when relating our multiplicity of RE results in the sequel to Schmitt-Grohe and Uribe ([70]). Our results do not require Armington aggregators for any of our theoretical results, rather simply preferences which satisfy Assumption 1.

[23] A useful fact concerning elements $C^T(S) \in \mathbf{C}^f$ is for any element $C^T \in \mathbf{C}^f$, $D'(S) = \kappa(Y^T + p(C^T(S))Y^{NT}$ is increasing in S. That will be a useful fact in our construction of RE.

[24] For latter reference, note $\mathbf{C}^f(\mathbf{S})$ is *not* a complete lattice.

[25] This is not trivial, as Schmitt-Grohé and Uribe ([70]) discuss in some detail, so we handle this issue systematically in the existence proofs related to this section in the appendix.

[26] This fact basically follows from the fact that in any RE, the household policy function for tradeable consumption must be continuous over individual states (by Berge's maximum theorem).

To handle this latter issue, we propose a two step fixed-point approach, where the fixed points of the "first step" guarantee all the needed continuity conditions for any RE policy function c^{T*} in its first two arguments, while the "second step" computes aggregate laws of motions (which parameterize the first step fixed point, as well as the household's collateral constraint) such the *equilibrium* collateral constraint at $C^{T*}(S)$ is consistent with equilibrium policy functions $c^{T*}(d, y, d, y, C^{T*}(d, y)) = C^{T*}(d, y)$. We discuss these issues in a moment in this section, and in detail in the appendix of the paper where our main existence theorem for RE is proven.

3.1 The Household's Dynamic Programming Problem in a RE

A household entering any period faces a fixed interest rate $R > 0$ such that $\beta R < 1$,[27] with a current level of individual household debt $d \in \mathbf{D}$, current realizations of endowments $y = (y^T, y^N) \in \mathbf{Y}$, and facing an aggregate economy in state $S \in \mathbf{S}$ whose continuation aggregate dynamics are parameterized by a single function $C^T \in \mathbf{C}^f(\mathbf{S})$.When entering the period in state $s = (d, y, S)$, the household then faces a feasible correspondence given by:

$$G(s; C^T) = \{c \in \mathbf{R}^2_+, d' \in \mathbf{D} | \text{ (14a) and (15) hold}\}$$

where

$$c^T + p(C^T(S))c^N \leq y - d + p(C^T(S))y^N + \frac{d'}{R} \tag{14a}$$

$$d' \leq \kappa(y^T + p(C^T(S))y^N) \tag{15}$$

which is a continuous correspondence in (d, y) for each $S \in \mathbf{S}$. In state $s = (d, y, S) \in \mathbf{D} \times \mathbf{Y} \times \mathbf{S}$, facing an aggregate economy characterized by a law of motion on per-capita debt D' using (13), a recursive representation of the household's sequential decision problem can be then be constructed as unique value function $V^*(s; C^T)$ solving a Bellman equation for each $C^T(S) \in \mathbf{C}^f$, $C^T(S) > 0$ for all $S \in \mathbf{S}$: [28]

$$V^*(s; C^T)) = \max_{x=(c^T, c^N, d') \in G(s; C^T)} U(c^T, c^N) + \beta \int V^*(d', y', Y', \Phi(S; C^T); C^T)\chi(dy') \tag{16}$$

Under Assumption 1(a-e), 1(g), and Assumption 2, for each $C^T \in \mathbf{C}^f$, $C^T(S) > 0$, standard arguments yield the following facts about solutions to equation (16).[29] First, noting the strict concavity of the primitive data under Assumption 1(g), the unique optimal policy function associated with the solution to (16) is given by:

$$\begin{aligned} c^*(s; C^T(S)) = \arg \max_{x=(c^T, c^N, d') \in G(s; C^T)} & U(c^T, c^N) \\ & + \beta \int V^*(d', y', Y', \Phi(S; C^T); C^T)\chi(dy') \end{aligned} \tag{17}$$

where by a standard application of Berge's maximum theorem to the right side of (16), the vector of consumption policies $c^*(s; C^T) = (c^{T*}(s; C^T), c^{NT*}(s; C^T))$ are jointly continuous in its first two arguments at $V^*(s; C^T)$, where $V^*(s, S; C^T)$ is continuous in (d, y), strictly concave and decreasing in d for each (y, S), and increasing in y, each (d, S).

[27]We shall only in this paper the case of RE for the more common case that $\beta R < 1$. The case of $\beta R = 1$ has been studied in the literature (e. g., Schmitt-Grohé and Uribe ([70]), but this case is delicate per the question of existence of SCE and/or RE. The problem is compactness relative to *stochastic* dynamics (as opposed existence of deterministic steady states). Similar issues arise when studying the case where $\beta R \geq 1$. In Pierri and Reffett ([60]), we consider both of these cases, and impose sufficient structure of SCE and RE to exist.

We should note, when studying the stochastic properties of the DSGE model under consider here in Section 5, setting $\beta R < 1$ is not a problem when fitting the model to the observations of Sudden Stops in the empirical literature.

[28]Note, when $C^T(D, Y)- = 0$ in any state, just set the value function $V^* = 0$ and the policy functions also all equal zero in all states (D, Y). This trivial case will be carefully handled in our contruction of a RE below.

[29]Notice, for this section, we do not need Assumption 1(f). That is, we can allow Inada conditions at $c_1 \to 0$. This means, for our arguments per characterizing RE, we allow for all standard utility functions (noting, that to meet assumption 1(e), we can always take period utility to be $U(c) = u(c) + \eta c$ for $\eta > 0$ and sufficiently small.

Second, relative to the primal dynamic programming problem in ((16)), by appealing to the duality results in Rincon-Zapatero and Santos (([62]), Proposition 3.1 and Theorem 3.1), one can prove there exists a well-defined recursive Lagrangian dual formulation of (16) that can be characterized as follows: for $C^T(S) > 0$, $C^T \in \mathbf{C}^f$, $c \in \mathbf{C} = \{c \in \mathbf{R}^2_+ | c^T \in [0, c^{\max}], c^N \in Y^N\}, d' \in \mathbf{D}$, we have:

$$v^*(s; C^T) = \inf_{\lambda, \mu \geq 0} \max_{c, d' \in \mathbf{C} \times \mathbf{D}} L(c, d', \lambda, \mu; s, v^*; C^T) \tag{18}$$

where, the recursive dual Lagrangian is given by:

$$\begin{aligned} L(c, d', \lambda, \mu; d, y, S, v^*; C^T) = U(c^T, c^N) + \beta \int v^*(d', y', \Phi(S; C^T), y; C^T)\chi(dy') \\ + \lambda\{y^T - p(C^T(S))y^N - \frac{d'}{R} - c^T + p(C^T(S))c^N\} \\ + \lambda\mu\{\kappa(y^T + p(C^T(S))y^N) - d'\} \end{aligned} \tag{19}$$

where under Assumption 1(a-e), 1(g), and 2, $v^*(s; C^T) = V^*(s, C^T)$, the recursive dual problem in (18) is well-defined, finite, strong duality holds with the Lagrangian dual value equal to the Lagrangian primal value, the Lagrangian dual solutions and values equal to those of the (16). Finally, this Lagrangian dual formulation in (19) admits a system of unique system of *stationary* KKT multipliers, $\lambda^*(s; C^T)$ and $\mu^*(s; C^T)$ associated with the infinite horizon sequential dual program that dualizes the household's sequential primal optimization problem in (4) from all initial conditions. Importantly, the associated KKT solutions $\{(\lambda^*(s; C^T), \mu^*(s; C^T)) ; (c^{T*}(s; C^T), c^{N*}(s; C^T), g^*(s; C^T))\}$ in (19) are the unique saddlepoints of (18) dual with value function $V^*(s; C^T) = v^*(s, C^T)$, with the envelope theorem for (16) in d given by:

$$\begin{aligned} \partial_d v^*(s; C^T) &= \lambda^*(s; C^T) \\ &= \partial_d V^*(s; C^T) \\ &= U'(A(c^{T*}(s; C^T), c^{N*}(s; C^T)))A_1(c^{T*}(s; C^T) \end{aligned}$$

where $(c^{T*}(s; C^T), c^{N*}(s; C^T))$ is the vector optimal solutions for consumption goods for the primal dynamic program in (16).

Using these facts, the system of first order conditions (necessary and sufficient) for our problem in (19) can be stated as follows: for stationary KKT multipliers $(\lambda^*(s; C^T), \mu^*(s; C^T))$, the optimal policy functions $((c^{T*}(s; C^T), c^{N*}(s; C^T))$, and $d'^*(s; C^T) = g^*(s; C^Y))$ satisfy the following:

$$\lambda^*(s; C^T) = U'(A(c^{T*}(s; C^T), c^{N*}(s; C^T)))A_1(c^{T*}(s; C^T) \tag{20}$$

$$p(C^T(S)) = \frac{A_1(c^{T*}(s; C^T))}{A_2(c^{N*}(s; C^T))} \tag{21}$$

$$\{\frac{1}{R} - \mu^*(s; C^T)\}\lambda^*(s; C^T) = \beta \int \lambda^*(d', y', \Phi(S; C^T), y'; C^T) \tag{22}$$

$$\{d' - \kappa\{p(C^T(S))y^N + y^T\}\mu^*(s; C^T) = 0, \ \mu^*(s; C^T) \geq 0 \tag{23}$$

where the law of motion on individual debt in (22) is given by:

$$d'^*(s; C^T) = \inf\{R\{c^{T*}(s, C^T) - y^T - p(C^T(S))y^N + d\}, \kappa\{y^T + p(C^T(S))y^N\} \tag{24}$$

and the law of motion on per-capita debt D' in (22) is given by $\Phi(S; C^T)$ in equation (13).

3.2 The Structure of RE Tradeable Consumption

When developing our RE operator in a moment, it proves very useful to first characterize how the structure of the optimal tradeable consumption policy varies over the two "regimes" (i.e., equilibrium states where the household is collateral constrained versus equilibrium states where household is not collateral constrained). To provide an answer to this question, first note for any household who enters any period in state $s = (d, y, S) \in \mathbf{S}$, after we impose the RE condition that $c^{*N}(s^e) = y^N = Y^N$, the household's budget constraint in (14a) is:

$$c^{*T} = y^T - d + \frac{d'^*}{R} \tag{25a}$$

If the per-capita tradeable consumption is given by $C^T \in \mathbf{C}^f$, the law of motion on optimal level of debt in equilibrium for the representative household is:

$$d'^*(s; C^T) = \inf\{R\{c^{T*}(s, C^T) - y^T + d\}, \kappa\{y^T + p(C^T(S))y^N\} \tag{26}$$

where $c^{T*}(s; C^T(S))$ is the household's optimal tradeable consumption. Using (20) into (22), we can then define the mapping :

$$Z_p^*(x, s; C^T) = \frac{U_1(x, y^N)}{R} - \beta \int U_1(c^{T*}(d', y', D', Y'; \; C^T)\chi(dy') \tag{27}$$

where the evolution of per-capita debt D' in (27) is given by equation (13), and the evolution of individual optimal debt for d' is given by (24). If we let $x^*(s; C^T)$ be the implicit solution of the following:

$$Z_p^*(x^*(s; C^T(S)), s; C^T(S)) = 0$$

which under Assumption 1(a)-(e), 1(g) and Assumption 2 is well-defined as a function continuous in (d, y), the implied optimal debt associated with a tradeable consumption plan $x^*(s; C^{T*})$ would be:

$$d_{x^*}(s; C^T) = R\{x^*(s; C^T) - y^T + d\} \tag{28}$$

If the debt level $d_{x*}(s, C^T)$ in (28) satisfies $d_{x*}(s; C^T) \leq \kappa\{y^T + p(C^T(S))y^N)\}$, the household is not debt-constrained in state $s = (d, y, S) \in \mathbf{S} \times \mathbf{S}$ for the aggregate tradeable policy $C^T(S)$, and the optimal policy for tradeable is given by: [30]

$$c_{uc}^{T*}(s; C^T) = x^*(s; C^T) \tag{29}$$

Alternatively, $d_{x*}(s, C^T) > \kappa\{y^T + p(C^T(S))y^N\}$, the debt constraint binds in this state s, and the optimal tradeable consumption is

$$c_c^{T*}(s; C^T(S)) = (1 + \frac{\kappa}{R})y^T - d + \frac{\kappa}{R}p(C^T(S))y^N. \tag{30}$$

In the latter case, the Euler equation in (20) binds, with $\mu^*(s; C^T) > 0$.

So the RE optimal policy for tradeable consumption has the following form for any $s = (d, y, S)$:

$$c^{T*}(s; C^T) = \inf\{c_{uc}^{T*}(s; C^T), c_c^{T*}(s; C^T)\} \tag{31}$$

where the infimum is computed at each (s, C^T) can be show to preserve *joint continuity* over individual states (d, y) for each S to $c^{T*}(s; C^T)$ by a standard application of the Berge's maximum theorem (noting the compactness of the state space). Additionally, one can show $c^{T*}(s; C^T)$ decreasing in d, increasing in y, each $S \in \mathbf{S}$.[31] We can use this structure of the optimal tradeable policy over the "two collateral constraint" regimes to develop our two step monotone approach to constructing RE in the next section.

[30] If if $d_{x*}(s; C^T) \leq \kappa\{y^T + p(C^T(S))y^N)$, this implies the collateral constraint is either slack or saturated (but not binding) at this state s. This, in turn, implies in equation (22) that the KKT multiplier on the debt constraint is $\mu^*(s; C^T) = 0$, and $Z_*^p(x, s; C^T)$ is the actual FOC for the household after imposing $c^{NT*} = y^{NT}$ in a RE.

[31] In the appendix, the proof of Lemma 6 actually proves these comparative statics of $c_c^{T*}(s; C^T(S))$ in (d, y) each $S \in \mathbf{S}$.

One last final note about the policy $c^{T*}(s;C^T)$. It can easily be shown that $c^{T*}(s;C^T)$ in (31) is *not* monotone increasing in $C^T(S)$ for each s. That is, the pecuniary complementarities induced by the equilibrium price-dependent collateral constraint *do not induce a monotone operator* directly via the household's policy function. Indeed, as we shall show in a moment, the requisite *single-crossing property* induced by pecuniary externalities that generates a global source of multiple RE in this model is an *equilibrium object,* and cannot be disentangle from the fixed point construction we shall use to construct the set RE itself.

3.3 A Multistep-Monotone Operator for Constructing RE

Motivated by the structure of the optimal tradeable consumption policy in equation (31), we now define our approach to constructing RE. The challenge of construction a RE operator models of Sudden Stops is because of collateral constraints are *price dependent,* they are *equilibrium objects.* To address this complication of structural change over different equilibrium "regimes" (e. g., collateral constrained states versus uncollateral constrained states), we construct an RE equilibrium consumption operator that models RE tradeable consumption as a "coupled" fixed point of a *two-step* operator that in its definition explicitly keeps track of the two equilibrium regimes that characterize the stochastic equilibrium dynamics of Sudden Stop models.

Intuitively, in the first step, we compute the RE tradeable decision rule *conditional* on holding the "pecuniary externality" *fixed* at the relative price of non-tradeable parameterized by tradeable consumption level $C^T(S)$. This first step mapping will turn out to be a monotone (order continuous) contraction in an appropriately define complete metric space, have a unique strictly positive fixed point for each fixed $C^T(S)$, and this strictly positive fixed point is continuous in the topology of pointwise convergence in $C^T(S)$.Then, in the second step, noting that this strictly positive fixed point in the first step is also isotone in $C^T(S)$, our second step mapping is order-continuous on its domain. Therefore, using a order-theoretic fixed point constructions we can compute the "least" and "greatest" RE collateral constraint consistent with this first step fixed point, which in turn induces the *set* of actual RE via its coupling with first stage fixed point).

Along these lines, image the structure of the (unknown) RE tradeable consumption "tomorrow" is given by the following mapping when $d = D$, and $y = Y$:

$$C(c, C^T)(d, y, S) = \inf\{c(d, y), C_c^T(D, Y, C^T(S))\} \tag{32}$$

Where $C_c^T = (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(D,Y)Y^N$, depends only on $C^T(S)$ the per-capital level of tradeable consumption. The mapping $C(c, C^T)(d, y, S)$ is constructed as the pointwise infimum of *two unknown functions*: (i) $c(d, y) \in \mathbf{C}^p(S)$, where $\mathbf{C}^p$ is the space of candidate functions for tradeable consumption in states where the household is "conditionally" not collateral constrained, and (ii) $C^T \in \mathbf{C}^*$ which RE tradeable consumption tomorrow when households are collateral constrained (where $\mathbf{C}^*$ will be defined below). Per the latter function, notice $C^T(S)$ parameterizes a "guess" at the *equilibrium* collateral constraint (and, hence the per-capital *equilibrium* collateral-constrained consumption $C_c^T(D, Y, C^T(S))$. Given this guess at "tomorrow's" tradeable consumption, we use the household's Euler inequality to compute the implied level of tradeable consumption "today" in equilibrium. We denote this implied mapping "today" by $A(c, C^T))(d, y, S)$ when $d = D$ and $y = Y$. It will true out then a strictly positive solution to the implied "coupled" fixed point problem will induce a RE for the economy over all equilibrium states.[32] The mapping $A(c, C^T))(d, y, S)$ will also turn out to be *jointly monotone* on its domain, with its domain being a complete lattice under pointwise partial orders. This implies powerful order theoretic fixed point methods can be brought to bear on this problem to characterize the existence and order structure of the set of RE, as well as the characterization of RE comparative statics.

[32]What will be critical in a moment, when defining our RE operator, is by imposing $d = D$, $y = Y$ in the definition of the operator, and computing tomorrow's consumption as $C(c, C^T)(d', y', d', y')$, the two equilibrium "branches" will be *coupled* by imposing equilibrium between transition structures for individual and aggregate state variables.

To formalize our construction, we begin by defining the first step domain of our two-step RE operator.[33] For the first step of our construction, we shall fix the "second step" function at $C^T \in \mathbf{C}^f$, and take the domain of our the operator $A(c;C^T)(d,y,S)$ the set of functions $c = c(d,y) \in \mathbf{C}^p(\mathbf{D}^e{\times}\mathbf{Y}^e)$, where:

$$\begin{aligned}\mathbf{C}^p(\mathbf{D}^e{\times}\mathbf{Y}^e) = \{c(d,y) | c \in \mathbf{C}^f, 0 \leq c(d,y) \leq y^T - d + (d^{Max}/R),\\ c(d,y) = \tilde{c}(d,y,d,y),\ \tilde{c}(d,y;d,y) \text{ decreasing in } d,\\ \text{increasing in } y,\\ \text{s.t. } (*) \text{ for a fixed } y,\ -d' = R(y^T - d - c(d,y,d,y)) \text{ decreasing in } d,\ \text{increasing in } y\}\end{aligned} \tag{33}$$

where d^{Max} is the maximal level of debt as shown in section 2, and we use the notation $s^e = (d,y,d,y) \in \mathbf{D}^e{\times}\mathbf{Y}^e$ to make clear the domain of any RE function is defined over a *diagonal* of individual state variable $s = (d,y,S)$:

$$\mathbf{D}^e{\times}\mathbf{Y}^e = \{(d,y) |\ (d,y) = (d,y,D,Y),\ d = D,\ y = Y\} = \mathbf{S}$$

where by construction $s^e \in \mathbf{D}^e{\times}\mathbf{Y}^e = \mathbf{S}^e$ is an *equilibrium* state of the household. We endow $\mathbf{C}^p(\mathbf{S}^e)$ with its relative pointwise partial order.

Notice a few facts about $\mathbf{C}^p(\mathbf{S}^e)$: (a) it is an equicontinuous collection of functions in the uniform topology (hence, compact), so each $c \in \mathbf{C}^p$ is continuous; (b) it is a nonempty subcomplete lattice of $\mathbf{C}^f(\mathbf{S})$ in its relative pointwise partial order; (c) for any function $c(d,y) \in \mathbf{C}^p$, the implied policy function for $d'(d,y)$ is *decreasing* in y, and *increasing* in d, so the elements of the space $\mathbf{C}^p(\mathbf{D}^e{\times}\mathbf{Y}^e)$ are consistent with the fact that when the household is *not* debt constrained, debt works as a consumption smoothing device for household relative to tradeable endowment shocks.

To define our RE operator, we first rewrite Z^*_p in (27) in a RE as follows: for any $c(d,y) \in \mathbf{C}^p$, when $c(d,y) > 0$, using $C(d,y;c,C^T(d,y))$ is defined in (32), imposing equilibrium between individual and aggregate states $d' = D'$ and $y' = Y$", we can define the mapping Z^*_{uc} is defined as follows:

$$\begin{aligned}Z^*_{uc}(x,s^e;c,C^T), = \frac{U_1(x,y^N)}{R} - \\ \beta \int U_1(C(R(x-y^T+d),y',R(x-y^T+d),y')\chi(dy')\end{aligned} \tag{34}$$

Where we denote the equilibrium state space by $s^e = (d,y,d,y) \in \mathbf{S}^e$. Relative to Z^*_p in (27), we have done the following: (a) assumed s^e is *not* collateral constrained, and (b) set $x = C^T(S)$ pointwise in equilibrium for the updating of the aggregate debt state in the it's aggregate law of motion. Of course, per (a), in a moment when defining the our RE operator, we will verify if this "assumption" is correct. Relative to (b), this is critical as this substitution is precisely what we need to model *equilibrium* single crossing between the household's tradeable consumption and the aggregate level of tradeable consumption.

To define then our RE operator, for any $c(d,y) \in \mathbf{C}^p, c > 0$, and $C^T \in \mathbf{C}^*(\mathbf{D}^e{\times}\mathbf{Y}^e)$, $Z^*_{uc}(x,s^e;c,C^T)$ is strictly decreasing in x, increasing in (c,C^T,y), and decreasing in d, and we can compute function $x^*_{uc}(s^e;c,C^T)$ implicitly in this expression:

$$Z^*_{uc}(x^*_{uc}(s^e;c,C^T),s^e;c,C^T) = 0 \tag{35}$$

which is well-defined as a function as Z^*_{uc} is strictly decreasing under Assumption 1, is continuous in its first two arguments, all the parameters, and pointwise continuous (topology of pointwise convergence) in (c,C^T) for each s^e,[34] and under Assumption 1 plus Assumption 2, the root $x^*_{uc}(s^e;c,C^T) > 0$. By a standard comparative statics argument, under Assumptions 1 and 2, $x^*_{uc}(s^e;c,C^T)$ is increasing in (c,C^T,y) and decreasing in d.

[33]Note, the definition of the second-step domain then will "couple" with the definition of the first step operator and its domain. For the moment, we simply take the second-step domain to have $C^T \in \mathbf{C}^f$ where $\mathbf{C}^f$ was defined earlier.

[34]For the situation that $c(d,y) = 0$ in any state, take $x^*_{uc}(s^e;c,C^T)$=0.

Then following the construction in the previous section for the optimal tradeable consumption policy, the implied debt level associated with the tradeable consumption $x^*_{uc}(s^e; c, C^{T*})$ will be:

$$d_{x^*_{uc}}(s^e; c, C^T) = R\{x^*_{uc}(s^e; c, C^T) - y^T + d\} \tag{36}$$

If the debt level $d_{x^*_{uc}}(s^e, c, C^T)$ in (36) satisfies $d_{x^*_{uc}}(s^e; c, C^T) \leq \kappa\{Y^T + p(C^T(S))Y^N\}$, the household is is not debt-constrained state $s^e = (d, y, d, y) \in \mathbf{S} \times \mathbf{S}$ for (c, C^T), [35] and:

$$A_{uc}(c; C^T)(s^e) = x^*_{uc}(s^e; c, C^T) \tag{37}$$

Alternatively, if $d_{xu}c^*(s, C^T) > \kappa\{y^T + p(C^T(S))y^N$, the debt constraint binds in state s, and the optimal tradeable consumption is

$$A_c(C^T)(S) = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N. \tag{38}$$

Then our operator for $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$ will be defined as:

$$\begin{aligned} A(c, C^T)(s^e) &= \inf\{A_{uc}(c, C^T)(s^e), A_c(C^T)(S)\} \text{ when } (c, C^T) > 0 \\ &= 0 \text{ else} \end{aligned} \tag{39}$$

We now construct the RE in two steps. In Lemma 6, we show for each fixed $C^T \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, there is a unique positive fixed point of partial mapping $A(c; C^T, \beta, \kappa, R)(s^e) \in \mathbf{C}^p$ (where in the Lemma, we now make explicit how the operator varies in the deep parameters (β, κ, R)) eventually RE comparative statics will be of interest in our main theorem).

Lemma 6 *Under Assumption 1(a-e), 1(g), and 2, for each $C^T(S) \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, and $S \in \mathbf{S}$, for the operator $A(c; C^T, \beta, \kappa, R)(s^e)$, (a) there exists a unique strictly positive fixed point $c^*(d, y, C^T(S), \beta, \kappa, R)$ in $\mathbf{C}^p(S)$; (b) this fixed point can be computed by the decreasing chain of successive approximations $\inf A^n(c_{\max}; C^T(S), \beta, \kappa, R)(d, y) \searrow c^*(C^T(S), \beta, \kappa, R)(d, y)$; (c) $c^*(C^T, \beta, \kappa, R)(d, y)$ is a monotone operator on $\mathbf{C}^f(\mathbf{D}^e \times \mathbf{Y}^e)$, (d) $c^*(C^T, \beta, \kappa, R)(s^e)$ is decreasing in (β, R), and increasing in κ.*

We now prove our main existence result in this section using the results in Lemma 6. We construct our second step operator using the first step unique strictly positive fixed point $c^*(C^T, \beta, \kappa, R)(d, y)$. The domain for second step mapping will be denoted $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$, and defined as follows:

$$C^T \in \mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e) = \{C^T(d, y) | C^T \in \mathbf{C}^f, 0 \leq C^T(d, y) \leq y^T - d + (d^{Max}/R), \tag{40}$$

$$C^T(d, y, d, y) = \inf\{c^*(d, y, C(d, y)),\ C(D, Y)\},\ d = D, y = Y$$

$$c(d, y; C(d, y)) \in \mathbf{C}^{p*}(d, y),\ C(D, Y) \in \mathbf{C}^f \text{ st } D'(d, y) = \kappa(y^T + p(C(d, y))y^{NT}\ \} \tag{41}$$

$$\subset \mathbf{C}^f(\mathbf{D}^e \times \mathbf{Y}^e) \tag{42}$$

The elements of the space $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e) \subset \mathbf{C}^f(\mathbf{D}^e \times \mathbf{Y}^e)$ have the exact same structure as those in $\mathbf{C}^p$ over *individual states*, but drops the condition (*) in (33) governing RE debt dynamics when collateral constrained binds. This reflects the fact that when the collateral constraint binds in equilibrium, the properties of the implied RE debt dynamics *reverse in order* to those implied by $c^*(d, y, C^T(S)) \in \mathbf{C}^p$ (the unconstrained regime for fixed C^T). We note $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ is a nonempty complete lattice under pointwise partial orders.[36]

[35] Notice, when the debt $_{x^*_{uc}}(s^e; c, C^T) = \kappa\{Y^T + p(C^T(S))Y^N\}$, the collateral constraint is saturated, but not binding. In this case, the implied KKT multiplier on the collateral constraint would be 0.

[36] If you fix (d, y), the collection $c^*(d, y; C(d, y)) \in \mathbf{C}^{p*}$ for $C \in \mathbf{C}^f$ forms an equicontinuous collection. Also, when $C \in \mathbf{C}^f$ such that the associated debt $D'(D, Y) = \kappa(Y^T + p(C^{T*}(D, Y))Y^{NT})$, $D'(D, Y)$ is also decreasing (resp., increasing) in D (resp., Y). That implies $C(D, Y) - D'(D, Y) = Y - D$ is decreasing (resp, increasing) in D (resp, Y) in decreasing, the resulting set is equicontinuous. This is sufficient for $\mathbf{C}^*(\mathbf{D}^e \times \mathbf{Y}^e)$ to be a complete lattice under pointwise partial orders.

To define the second step operator we use the unique positive fixed point of the first step operator, and define the following nonlinear operator on the space $\mathbf{C}^*$: when $(d, y) = (D, Y) = S$:

$$A^*(C^T; \beta, \kappa, R)(s^e)) = C(c^*(C^T),\ C^T; \beta, \kappa, R)(s^e) \qquad (43)$$
$$= \inf\{c^*(C^T; \beta, \kappa, R)(s^e),\ A_c(C^T; \kappa, R)(s^e)\}$$

where we now make the dependence of the operator A^* and $C(c^*(C^T), C^T; \beta, \kappa, R)$ on the deep parameters (β, κ, R) explicit. We now have the following result.

Theorem 7 *Under Assumption 1(a-e), 1(g), and 2, there exist (i) a nonempty complete lattice $\Psi^*(R, \kappa, \beta)$ of RE in C^*, and (ii) the least and greatest elements of $\Psi^*(R, \kappa, \beta)$ are increasing in κ, and decreasing in (R, β), (iii) the least and greatest RE can be computed by successive approximations: i.e.*

$$\inf_n A^{*n}(0; R, \kappa, \beta) \nearrow \wedge\Psi^*(R, \kappa, \beta)$$
$$\sup_n A^{*n}(c_{\max}; R, \kappa, \beta) \searrow \vee\Psi^*(R, \kappa, \beta)$$

We conclude this section by relating our RE results on least and greatest RE to the multiplicity results in Schmitt-Grohé and Uribe ([69], [70]) relative to SCE near (deterministic) steady-states. What is clear from our approach is that relative to the set of RE, the issues raised in Schmitt-Grohé and Uribe ([69], [70]) are *global.* That is, in *any state* where collateral constraints bind (not just near steady state), in the stochastic RE (not just the deterministic version of the model), multiplicities of RE are possible (and in particular, “low” and “high” borrowing equilibria are possible).

More specifically, Schmitt-Grohé and Uribe ([70]) show near a *deterministic* steady-state of the this model, multiple SCE can exist as the SCE dynamical system for debt $(D_{t+1}) = \Psi^*(D_t)$, that mapping Ψ^* can have multiple fixed points near the steady-state. That is, they give sufficient conditions on primitives such that there exists a second equilibria of Ψ^* that is a type of “self-fulfilling” low borrowing equilibria, which can occur when there exists a state $D^* < D^{ss}$ where D^{ss} is the deterministic steady state, where people are “pessimistic” about the level of borrowing and aggregate tradeable consumption at debt-level D^*. In this case, if the collateral constraint binds at D^*, and if $c^{T*} = C^{T*} < C^{T,ss}$ is the level of tradeable consumption at $d^* = D^*$ for the household in a SCE, one can obtain an implicit equilibrium relation (written here in terms of C^{T*}) that has at D^*:

$$Z_c^*(C^{T*}, y; \kappa/R, D^*) = C^{T*} - (y^T + y^{NT}p(C^{T*}))(1 + \frac{\kappa}{R}) - D^* = 0$$

This solution for C^{T*} implies a (self-fulfilling) equilibrium with low debt and low tradeable consumption. An important feature of this argument is it is local and based upon *deterministic* perfect foresight equilibrium constructions. One should note, this argument *need not imply* multiple *stochastic* sequential competitive equilibria as in our main theorem relative to the set of SCE that are induced by RE.

But by a simple argument, we can now see the ideas of Schmitt-Grohé and Uribe ([70]) are actually *robust and global* for set of *stochastic* RE using the results in Theorem 7. To see this, in any state equilibrium state s^e where the collateral constraint binds, we have $A^*(C^T)(s^e)$ given by

$$A^*(C^T)(s^e) = A_c(C^T)(s) = (y^T + y^{NT}p(C^T(S)))(1 + \frac{\kappa}{R}) - d$$

Consider the following mapping when $d = D$, $y = Y$:

$$Z_{cc}^*(x^*(d, y, S, C^T),\ C^T(S); d, y, S) = x^*(d, y, d, y, C^T) - (y^T + y^{NT}p(C^T(S)))(1 + \frac{\kappa}{R}) - d = 0$$

where the subscript indicates this an equilibrium state that is collateral constrained, and where in this expression, we have $A_c(C^T)(s^e) = x^*(d, y, d, y; C^T)$ (which corresponds with our fixed point operator $A^*(C^T)(s^e)$ in a state s^e where the collateral constraint binds). Evaluating Z_{cc}^* at a fixed point of

$A_c(C^T)(s^e) = x^*(d,y,d,y) = C^{T*}(d,y)$ when $d = D$ and $y = Y$, we see unless the mapping $p(\cdot)$ and parameters κ, R, and y^{NT} are such that

$$Z_{cc}(x,x;y,\kappa/R,d) = x - (y^T + y^{NT}p(x))(1+\frac{\kappa}{R}) - d = 0$$

has multiple roots $x^*(d,y,d,y) \geq 0$, there will be *multiple RE* at this state s^e that satisfy the mapping Z

$$\overset{*}{_c}(x^*(d,y,d,y), C^{T^*}(d,y); d,y,d,y) = 0$$

for $x^*(d,y,d,y) = C^{T*}(d,y)$.[37] As mapping $A_c(C^T)(s^e)$ is also monotone, the least and greatest RE will exist such state s^e and be *ordered.* That is, if we define the correspondence

$$X^*(d,y,d,y) = \{x^*(d,y,d,y \geq 0|_{cc}(x,x;y,\kappa/R,d) = 0\}$$

which is the set of roots of $Z_{cc}(x,x;y,\kappa/R,d) = 0$ at equilibrium state s^e, this correspondence will have a least and greatest element (as the correspondence $X^*(d,y,d,y) \subset \mathbf{R}_+$ a chain, and $X^*(d,y,d,y)$ is nonempty and compact-valued under assumption 1 and 2 (and hence has a least and greatest element). Notice this also implies in equilibrium states s^e where the collateral constraint binds, as collateral constraints are price-dependent and $p(C)$ is increasing in C, the equilibrium collateral constraints will be *ordered* relative to least and greatest RE tradeable consumption levels. So as Schmitt-Grohé and Uribe ([70]) suggest, in general, we will have (globally) "low borrowing" (associated with "least" RE tradeable consumption) and "high borrowing" (associated with "greatest" RE tradeable consumption) in any equilibrium state s^e, and these least and greatest RE will be distinct in states where the equilibrium collateral constraint binds and the correspondence $X^*(d,\, y,\, d,\, y)$ is not a single-valued.

4. Generalized Markov Equilibrium

The previous section proves the existence of a GME representation of SCE that is time-invariant and defined on a *minimal state space.* The difference between GME representations of SCE and RE can now be explored, and will have important implications from a numerical perspective. GME representations of SCE differ substantially from those in section 3, and can be said though of as all the SCE that admit recursive representations as in Duffie, et. al. ([27]) or Feng, et. al. ([29]). As minimal state space recursive equilibria is a smaller set, it is possible to compute it efficiently. The later more general GME representations require more elaboration, which we will not discuss in this section of the paper.

Additionally, while the results in section 3 can be used to numerically approximate the SCE and to perform accurate numerical comparative statics exercises, we have been silent about simulations in these class of models. As discussed in Santos and Peralta- Alva ([63]), the starting point of any simulation experiment for a stochastic dynamic equilibrium model is an appropriate *stochastic steady state* notion. The most frequent stochastic steady notion used in the literature is an *invariant measure* (IM) for *some* recursive representation of the sequential equilibrium. Heuristically, an IM gives a sense of *probabilistic time invariance.* That is, if $\{x_t\}$ is a sequence of random variables generated from some Markov process and x_t is distributed according to an IM μ, then x_τ will be distributed according to μ for $\tau > t$.

This section introduces the notion of Generalized Markov Equilibrium (GME), which has an expanded state space when compare to the results in section 3. This equilibrium is a slightly modified version of the one in Feng, et. al. ([29]). Moreover, a GME is defined using the set of equations characterizing the SCE. Thus, the additional state variable and the direct connection with the SCE brings more memory into the model at the cost of allowing additional sources of equilibrium multiplicity. This paper proves that it is possible to refine the equilibrium set in a GME by picking a selection which insures the existence of an IM. In this sense, we are imposing long-run restrictions in order to refine the equilibrium set.

[37] Schmitt-Grohé and Uribe ([70]) give a *local* sufficient condition near the deterministic steady-state for this to be case for the case that the utility aggregator $A(c^T, c^{NT})$ is an Armington aggregator and near a steady-state. But clearly, their idea about the source of multiplicity applies in *any* equilibrium state s^e. That is, generally $Z_{cc}(x,x;y,\kappa/R,d)$ is not either strictly increasing or decreasing in x at each s^e under Assumption 1 (hence, roots are unique)

Provided a stationary recursive representation, the existence of an IM implies that simulations obtained from the SCE can be approximated by a time invariant and finite set of functions, abstracting from numerical errors. [38] This is possible by means of a *law of large numbers* which in turn requires the IM to be *ergodic*. From a practical perspective, ergodicity insures roughly speaking that "averages converges". That is, the existence of a stationary recursive representation, as the one derived in section 3 for instance, is not enough to insure the desired convergence, which is typically obtained using a law of large numbers. The ergodicity of the IM guarantees that the Cesaro average of any simulation starting from a "nice" initial condition will converge to an *expected value* computed using the stochastic steady state distribution. This last fact allows to connect the model with observed (time independent) stylized facts.

As discussed in Pierri ([58]), existence of an IM and its ergodicity in this type of model are related to the cardinality of the set of exogenous shocks. We prove the existence of an ergodic IM for economies with a *finite set of shocks* under milder assumptions than those studied in Pierri ([58]). This is possible because of the monotonicity properties of the (minimal state space) RE in endowment shocks combined with an occasionally binding (collateral) constraint. Under this setting it is possible to show that the model described in section 2 has an "irreducible atom", which is used in the stochastic process literature to prove the existence of an IM (see for instance Meyn and Tweedie, section 10). The ergodicity of this measure follows from its uniqueness. While these results are enough to insure convergence in the sense implied by a law of large numbers, finding the appropriate set of initial conditions maybe problematic as the process may have divergent paths. Fortunately, we can characterize the set of appropriate initial conditions and at the same time prove the existence of an ergodic IM.

4.1 A convenient recursive representation

In this subsection, we derive the set of Generalized Markov Equilibrium (GME). As this equilibrium have a bigger state space when compared with the recursive representation presented in section 3, it is more flexible (see Kubler and Schmedders ([38] for a discussion). We are interested in preserving the structural properties (i.e. differentiability of the value function, etc.) proved before as they will be useful to derive the results in this section. One of the purposes of writing the minimal state space equilibrium is to refine all possible GME in order to build a selection which replicates the observed behavior. It turns out that, if we restrict attention to a finite set of shocks, it is easy to characterize a "regeneration point" for the global stochastic dynamics in the model using the GME. This is a first step in order to find a recurrent structure that is robust to the presence of multiple equilibria, which typically generate discontinuous selections. It will only be possible to prove the existence of an invariant measure using some of the results derived for the minimal state space equilibrium (RE) but applied to Feng, et. al.'s representation. The properties of the endogenous variables derived for the optimization problem in the RE for the unconstrained case (i.e. when the collateral restriction does not hold with equality) will be useful to construct trajectories with positive probability which can be used to prove ergodicity. This is possible as any sequential equilibrium must be optimal given prices for the household.

As can be seen in the appendix for section 2, given assumption 1 and the results in theorem 1, any SCE can be characterized using a set of primal first order conditions (as inequalities) which do not depend on Lagrangian multipliers. The usefulness of this representation will be clear in this subsection. Moreover, the existence of well behaved envelopes for the value function in the RE implied that a SCE can be characterized recursively by the following equations:

$$[-A1(c_1)p + A_2(c_2)] = 0 \tag{44}$$

$$[\kappa\{y^T + py^N\} - d_+][U'\{A_1(y^T + d_+R^{-1} - d)\} - E(m_+)] = 0 \tag{45}$$

[38]As pointed out in Santos and Peralta Alva ([63]), truncation and interpolation errors could accumulate over time if they are not "controlled".

where $m = \frac{\partial V}{\partial d}$ is the envelope of the value function in the household's problem of the minimal state space representation of SCE, with $U' \equiv U'(A(y^T + R^{-1}d_+ - d; y^N))$. Given the compactness of the equilibrium set, the results in Feng, et. al. imply that equations (44) and (45) can be used to derive a correspondence, Φ, the so-called *equilibrium correspondence*, which contains the entire set of GME representations of SCE, where $\Phi : Z \times Y \longmapsto Z$ with $z = [d \quad y^T \quad y^N \quad c_1 \quad c_2 \quad p \quad m]$ with $z \in Z$, $y^T \in Y$ and Z compact.

Notice, we are restricting m to be an envelope of the value function from some RE. This restriction is required to show ergodicity as the proof involves a path for d_+ with some qualitative properties which follow directly from the optimization problem in the RE. Moreover, the results in section 3 insure that equation (45) under this restriction can be used to characterized any SCE. Thus, when the (collateral) constraint hits, we know that $U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq E(m_+)$ and the equilibrium for any given period can be computed using the following set of equations:

$$p = \frac{A_2(y^N)}{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)} \tag{46}$$

$$c_1 = y^T + R^{-1}\kappa\{y^T + py^N\} - d \tag{47}$$

$$c_2 = y^N \tag{48}$$

$$m = U' A_1(c_1) \tag{49}$$

$$U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - d)\} \geq E(m_+) \tag{50}$$

Given (y^T, d), all the remaining variables in z can be computed using (46) to (49) as long as $U'\{A_1(c_1)\} \geq E(m_+)$. That is, any $d \geq \overline{d}$ can be used to compute z with $U'\{A_1(y^T + R^{-1}\kappa\{y^T + py^N\} - \overline{d})\} = E(m_+)$.

As the recursive equilibrium notion in Feng, et. al. is computed "backwards" (i.e. given "z_+" you obtain z), the Euler equation imposes a looser restriction to the system when compared to the non-binding case. This fact turns out to be very useful to prove the ergodicity of the process in the finite state space case. We are now in position to formally define a GME.

Definition 8 *Generalized Markov Equilibrium (GME): Given a compact set Z which contains any state z_s that solves* (46) *to* (50) *backwards with $s = 0, ..., t+2$, the correspondence Φ mapping $Z \times Y \to Z$ can be defined as follows: take the sequence of backwards solutions $\{z_0, z_1, ..., z_{t+2}\}$. Let $L = 0$ be the system formed by equations* (46) *to* (50)*. Note that for a given vector $d_{t+3}(y_{t+2})$ with $y_{t+2} \in Y$, any pair (z_s, z_{s+1}) satisfies: $L(z_s, z_{s+1}; d_{t+3}) = 0$. Then, $z_{s+1} \in \varphi_{t+3}(z_s)$ where $\varphi_{t+3} \in \Phi$ is the selection associated with $d_{t+3}(y_{t+2})$. We will say that φ_{t+3} is a GME. If the vector $d_{t+3}(y_{t+2})$ can be chosen independently of t, we will say the the GME is stationary* [39] *$\varphi \in \Phi$. Let (Z, P_φ) defines a stationary Markov process with kernel given by P_φ. If (Z, P_φ) has an ergodic invariant* [40] *measure, we say that φ is ergodic.*

Given the compactness of Z, as there are a finite number of shocks, the measurability requirements for P_φ follows from Feng, et. al. ([29]). In section 5, we compute the 3 classes of selections and simulate the model for a standard set of parameters.

[39]We will refer to φ_{t+3} as a "non-stationary" equilibrium. Note that this selection generates a stationary process but it is defined around a particular event, associated with a given time period $t+3$. Stationary and ergodic selections are constructed with respect to a stationary object, defined by $L(z, z_+; d_{++}) = 0$. See the supplementary material for section 5.3, in the appendix, for a concrete example of this 3 types of selections.

[40]The results in this paper do not allow to distinguish between invariant and ergodic measures, so we will simply refer to the stochastic steady state as ergodic.

4.2 Stationarity and Ergodicity under a finite set of shocks

We now derive the set of stochastic steady state/stationary equilibria for the model. Formally, we show that Φ has an ergodic selection. It turns out that if we restrict the number of possible distinct values that y^T can take to be finite, we can prove the existence of an ergodic probability measure. In this framework, equation (50) can be used to construct a *point* d_* that generates a set which the process hits with positive probability starting from any initial condition. This point will be called *atom* and solves the system of equations given by (46) to (50) for wealth d_*, with the Euler equation holding with equality and for the lowest possible level of y^T. Moreover, the process will hit the atom in finite time. Thus, it creates an orbit which endows the dynamical system with a recurrent structure, which in turn implies that there will be a unique (and thus ergodic) invariant measure for each atom.

Once we find d_*, we will construct a stable state space. That is, any meaningful (i.e. with positive measure) subset of this state space will be hit by the process in finite time. This property, called irreducibility, will insure the uniqueness and ergodicity of the process. The existence of 2 regimes, one defined for equations (44) and (45) when the collateral constraint does not bind and the other given by equations (46) to (50), together with the possible multiple solutions to equation (46) suggests the presence of multiple possibly discontinuous Markov equilibria. If we allow for discontinuous selections of the equilibrium correspondence in the GME, we can construct a transition function that "jumps" to the atom every time the collateral constraint is hit, generating a "crises". Thus, the presence of multiple equilibria, which is in part a consequence of the long-term memory inherited from the SCE, and its implications for the smoothness of the selections $\varphi \in \Phi$ increases the predictive power of the model in the sense that it allows a better match of long run empirical regularities. Thus, it is critical to understand the "anatomy" of the equilibrium set, something that is done in section 5.1. As suggested by Stokey, Lucas and Prescott ([74]) [41], the existence of such a point is enough to derive a stationary Markov process. The results in Meyn and Tweedie ([52]) give us the tools to prove all the intermediate steps required to go from the existence of an stationary selection of the equilibrium correspondence to its ergodicity.[42] Our identification of an "atom" in this paper is related to the presence of occasionally binding constraints in non-optimal general equilibrium economies, and our application of these tools is novel in the literature. Thus, we will prove the results step by step as it is immediate to extend the methodology used in this paper to another related frameworks with equilibrium collateral constraints.

In any RE, the model is characterized using the Markov kernel and, thus, in 1 "step". This is a consequence of the "short memory" approach in this type of equilibrium. The GME, as it is computed directly from the sequential equilibria, allows us to bring "memory" into the picture. That is, it is possible to construct a *finite time path from the sequential equilibria*, which insures that the model will match the observed stylized facts, and to choose the appropriate selection from the equilibrium correspondence, that guarantees the existence of an ergodic Markov process. Using the former we obtain a short term match with data and because of the latter we can tie the deep parameters of the model to the long run behavior of the same observed time series. In other words: we show that there exist a selection which insures ergodicity and it is also compatible with the presence of multiple different "phases" of crises, which may have variable time spells. Section 5.2 illustrates how to use an ergodic GME to match the anatomy of a "typical" sudden stop.

Let us start by formally defining an "accessible atom", which can be thought as a point that is non-negligible from a probabilistic perspective and gets "hit" frequently. Let $\varphi \sim \Phi$ be a selection of the equilibrium correspondence defined in the previous section. The compactness of $Y \times Z$ and Z guarantees the measurability of φ. [43] Further, $P_\varphi(z, A) \equiv \{p(y^T \in Y : \varphi(z, y^T) \in A)\}$ defines a Markov operator (i.e. $P_\varphi(., A)$ is measurable and $P_\varphi(z, .)$ is a probability measure) and (Z, P_φ) a Markov process where y^T is assumed to be iid with probability $p(y^T)$. [44] Let $P^n_\varphi(z, A)$ be the probability that the Markov chain goes from z to any point in A in n steps with A being measurable, let ψ be some measure, and $B(Z)$ be the Borel sigma algebra generated by Z. Then the set $A \in B(Z)$ is *non-negligible* if $\psi(A) > 0$.

[41] See for instance exercise 11.4, Ch. 11.

[42] See Meyn and Tweedie ([52]), chapters 5, 8 and 10 for a detailed discussion of the implications of the existence of an atom for the existence of an invariant probability measure.

[43] for example, Stokey, Lucas and Prescott, ([74]), Th. 7.6, p. 184).

[44] For example, see Grandmont and Hildenbrand ([34]).

A chain is called *irreducible* if, starting from any initial condition, the chain hits all non-negligible sets with positive probability in finite time (i.e. $\psi(A) > 0 \longrightarrow P^n_\varphi(z, A) > 0$. [45] Intuitively, irreducibility is a notion of connectedness for the Markov process as it implies non-negligible sets are visited with positive probability in finite time.

We are now in position to define an atom and state an important intermediate result.

Definition 9 *Accessible Atom. A set $\alpha \in B(Z)$ is an atom for (Z, P_φ) if there exists a probability measure ν such that $P_\varphi(z, A) = \nu(A)$ with $z \in \alpha$ for all $A \in B(Z)$. The atom is accessible if $\psi(\alpha) > 0$.*

Intuitively an atom is a set containing points in which the chain behave like an iid process. Any singleton $\{\alpha\}$ is an atom. Note that there is a trade off: if the atom is a singleton, the iid requirement is trivial but, taking into account that the state space is uncountable, the accessibility clause becomes an issue as it is not clear how to choose ψ. For instance, the typical measure, the Lebesgue measure, it is not useful as any singleton has zero measure in it. The same happens with irreducibility: when the state space is finite, it suffices to ask for a transition matrix with positive values in all its positions. In the general case, we need to define carefully what is a meaningful set as it is not possible to list all of them. Fortunately, when the state space Z is a product space between a finite set (Y) and an uncountable subset of $\Re^m$ there is a well know results that help us find an accessible atom in an irreducible chain (for proof, see Proposition 5.1.1 in Meyn and Tweedie.)

Proposition 10 *Suppose that $P^n_\varphi(z, \alpha) > 0$ for all $z \in Z$. Then α is an accessible atom and (Z, P_φ) is a $P_\varphi(\alpha, .)-$irreducible.*

Note, proposition 10 follows directly from standard results in Meyn and Tweedie, ([52]) where atom is implicitly assumed to be a singleton and the reference measure may be different from ψ which is called "maximal". Fortunately, if the chain is irreducible with respect to some measure, say $P_\varphi(\alpha, .)$, then it can be "expanded" to ψ (e.g, see Meyn and Tweedie, ([52], Proposition 4.2.2).

In order to apply Proposition 10 to our model, the finiteness of Y and the definition of the Markov kernel P will be critical. As we are considering a point, in order to show that $P^n_\varphi(z, \alpha) > 0$, it suffices to find a finite sequence $\{y_0, ..., y_n\}$ such that $\{\alpha\}$ is a solution to equations (46) to (50), the system associated with a binding collateral constraint. We want to associate the atom with an economic crises as this fact that will allow us to connect the invariant measure with a sudden stop (SS) or, equivalently, to relate the steady state of the model with the frequency of crises.

The power of an atom for characterizing the behavior of the Markov chain is well-known.[46] However, it is possible to illustrate the effect of an atom in the recurrence structure of the chain, which is critical to define an invariant measure (i.e. a measure μ which satisfy $\mu = \int P_\varphi(z, A)\mu(dz)$). Suppose that the atom is hit for the first time with positive probability in period $\tau_\alpha < \infty$ starting from z_0. Then, it is possible to define a (not necessarily probability) measure μ which gives the expected number of visits to a particular set in $B(Z)$, called it A, before τ_α. Stated differently, $\mu(A)$ gives the sum of the probabilities of hitting A *avoiding* α. Imagine the system in period $\tau_\alpha - 1$ is starting from z_0. Remarkably, when you "forward" μ 1 period (i.e. by applying the Markov operator to it, $\int P_\varphi(z, A)\mu(dz)$) the expected number of visits to A avoiding α is the same as the chain will hit α in period $n = \tau_\alpha$. Thus, μ must not change or equivalently $\mu = \int P_\varphi(z, A)\mu(dz)$. That is, μ is an IM. Provided that $\tau_\alpha < \infty$, it is possible to normalize μ to be a probability measure, which we will call π (this property is called "positivity"). Further, as the

[45] e. g., see Meyn and Tweedie (([52], proposition 1)

[46] Meyn and Tweedie mention the importance of an atom for general state space Markov chains relative to countable state space Markov chains (e. g., [52], p96). A discussion of the important of the existence of an atom in the context of the Markov chain theory is outside the scope of this paper, but a systematic discussion of this fact is presented in Meyn and Tweedie (Chapters 8, 10 and 17).

accessibility of the atom comes together with the irreducibility of the chain (see proposition 1), it is not surprising that the IM is unique as the chain does not break into different "unconnected islands". Finally, the Krein-Milman theorem insures the ergodicity of the chain provided its uniqueness (see Futia, ([30])).

Now, in order to connect the existence of an invariant measure with the frequency of crises let τ_α be the time when the process hits the collateral constraint. Then, $\mu(A)$ gives the cumulative probability of hitting A *avoiding* a crises. Thus, the frequency of a SS affects the stationary distribution μ. Frequent crises implies more volatility. To understand this relationship for equilibrium paths in the steady state, note that every time the process hits the collateral constraint it reverts to α. Because $P_\varphi(\alpha, A) = \nu(A)$, the value of $z_{\tau_\alpha+1}$ is *independent* of the past, which implies that it loses all the inertia inherited from the Markovian structure of the process. Thus, the equilibrium stochastic dynamics behave unconditionally with respect to the past, increasing its variability.

To relate these facts with the empirical performance of the model, we need a Law of Large numbers. As the measure is ergodic, it is well known that $\sum_{t=0}^{T}(z_t/T) \to E_\mu(z)$ almost everywhere, where $\to$ means T tends to infinite. [47] In other words, the existence of an ergodic measure insures that a sample mean computed by an increasing large time series of simulated data will hit the steady state of the model, represented by the mean $E_\mu(z)$, for a large fraction of possible paths $\{z_t\}_{t=0}^{\infty}$. Thus, the time spells without a SS shape the long run distribution of the model, affecting its ability to replicate stylized facts. The following theorem proves the existence of an ergodic measure μ for the model described in section 2.

Theorem 11 *There exists an $\varepsilon > 0$ such that $y_{lb} \in (0, \varepsilon)$, a compact set $J_1 \subseteq Z$ with $\Phi : J_1 \times Y \longrightarrow J_1$ and a selection $\varphi \sim \Phi$ such that the process defined by (J_1, P_φ) has an unique ergodic probability measure.*

5. Applications

We now apply the results obtained in sections 3 and 4 to characterize the model's SCE described in section 2. It is organized in 3 subsections: a characterization of the equilibrium set which focus on multiplicity, a qualitative description of short term equilibrium dynamics together with its connection with the anatomy of observed crises and a quantitative exploration of short and long run simulations. In this last subsection, we describe the algorithms generated by the Generalized Markov Equilibria (GME) and then use them to compute and simulate an ergodic, a stationary and a non-stationary equilibrium.

Since Blume ([17]) the literature recognizes a trade off between the multiplicity of sequential equilibria and the smoothness and continuity properties of the associated recursive representation as the presence of multiple equilibrium can generate discontinuities in minimal state space transition functions. The usefulness of the methods developed in sections 3 and 4 comes into play in the presence of this type of discontinuities, which may affect the existence of an ergodic equilibrium. Both RE and GME are robust to the presence of non-smooth or even discontinuous selections. Using the GME, we can add memory to the analysis, which is deeply connected with the presence of multiple equilibrium, and at the same time find a well behaved stochastic steady state. Thus, it is critical to investigate if the multiplicity of equilibrium is a serious issue.

Section 5.1 takes the model in section 2 and shows that it is possible to obtain multiple equilibrium when the collateral constraint binds for a *wide range of parameter values.* We show that the number of possible exogenous shocks is critical to generate multiplicity. To our knowledge, this is the first attempt to address this issue in a *stochastic* setting.[48] The type of models described in section 2 are typically used to designed macro-prudential policies that prevents the occurrence of SS, see Bianchi, ([12]). This is desirable because, following for instance Mendoza and Smith ([48]), it is possible to define this event as a sharp reduction in consumption caused by a reversion in the capital account surplus. By its own nature, a SS suggests the presence of a discrete jump in the endogenous variables of the model; a fact

[47] e. g., see Stokey, Lucas and Prescott, ([74]), chapters 11 and 12)

[48] See Schmitt-Grohé and Uribe ([70]) for a discussion of the non-stochastic steady state of the economy with local sunspots.

that is at odds with a continuous Markov equilibrium. The literature has solved this issue by imposing a discrete jump in the maximum debt-to-GDP ratio κ when the collateral constraint binds (see Seoane and Yurdagul ([65]), for a detailed discussion). This type of strategy suggests that the theory is insufficient to explain the occurrence of a SS as the model requires an unexpected change in the parameters to match the observed behavior.

On the contrary, by allowing the presence of multiple equilibrium and using a GME representation, we show in section 5.2 that a canonical SS model is capable of generating a collapse in foreign borrowing without imposing a change in the parameters or a "big negative shock" and at the same time preserves the desired ergodic properties of the recursive equilibrium. Further, this section also shows that the model generates a "Fisherian deflation", a term which is often used to describe a monotonically decreasing path of debt, consumption and real exchange rate associated with a downturn in credit conditions, caused by a tightening in the collateral constraint.[49] Thus, ergodic GME representations are capable of replicating the anatomy of a balance of payment crises and at the same time match the long-run behavior of the economy using an ergodic selection.

Given the existence of an appropriate solution algorithm, sometimes the literature only computes the dynamics associated with a Ramsey optimal taxation problem, which can be implemented as a solution to the model described in section 2, distorted with taxes (see Mendoza and Smith ([48])). Using such solution methods, it is not possible to keep track of the decentralized SCE. In other cases, a minimal state space recursive representation is imposed for the decentralized SCE and then one computes using a seemingly appropriate procedure (see, for instance, Bianchi, ([12])). Unfortunately, it is not clear if these algorithms will converge to an equilibrium transition function.

Section 5.3 presents the outline of an algorithm to compute several selections of a GME which does not suffer from these aforementioned problems. In this sense, we present a global algorithmic solution to the presence of multiple equilibrium and memory. We compute an ergodic, a stationary and a non-stationary selection. We find that, for a standard parameter set in the literature, the ergodic selection generates smoother consumption paths when compared with a stationary equilibrium. Equivalently, market incompleteness is less severe in the first type of equilibrium. The intuition is connected with the presence of an atom, which acts as an orbit for the dynamical system. Consequently, divergent paths are rule out in this equilibrium due to its recurrent structure; implying smoother paths for the same frequency of crises.

5.1 Multiplicity of equilibrium

From the discussion above, it is clear that the GME (and therefore, the SCE) generate multiple selections, each of them associated with a different envelope, m_+. In order to study this type of multiplicity, we must restrict attention to the subset of compact SCE as otherwise the set of GME could be empty. Thus, the bounds on marginal utility in assumption 1 are critical. However, there is another source of multiplicity, frequently studied in the literature (see Schmitt-Grohé and Uribe ([70])), which is concerned with the possibility of having more than 1 solution to equation (46) when the collateral constraint is binding. In order to handle this sort of multiplicity, the bounds on marginal utility are not required. Thus, as this section is concerned with this type of multiple equilibrium, we will use standard CES preferences as, for instance, in Bianchi ([12]).

We restrict attention to the source of multiplicity (and discontinuity) associated with the price dependent collateral constraint. Thus, we will focus only in the system generated by equations (46) to (50). Let $U(A(x)) = (A^{1-\sigma} - 1)/(1 - \sigma)$, $A(c) = (ac_1^{1-1/\xi} + (1 - a)c_2^{1-1/\xi})^{1/(1-1/\xi)}$ with $\sigma = 1/\xi = 2$ and $a = 1/2$. We will assume that Z is compact. Then, equation (46) becomes:

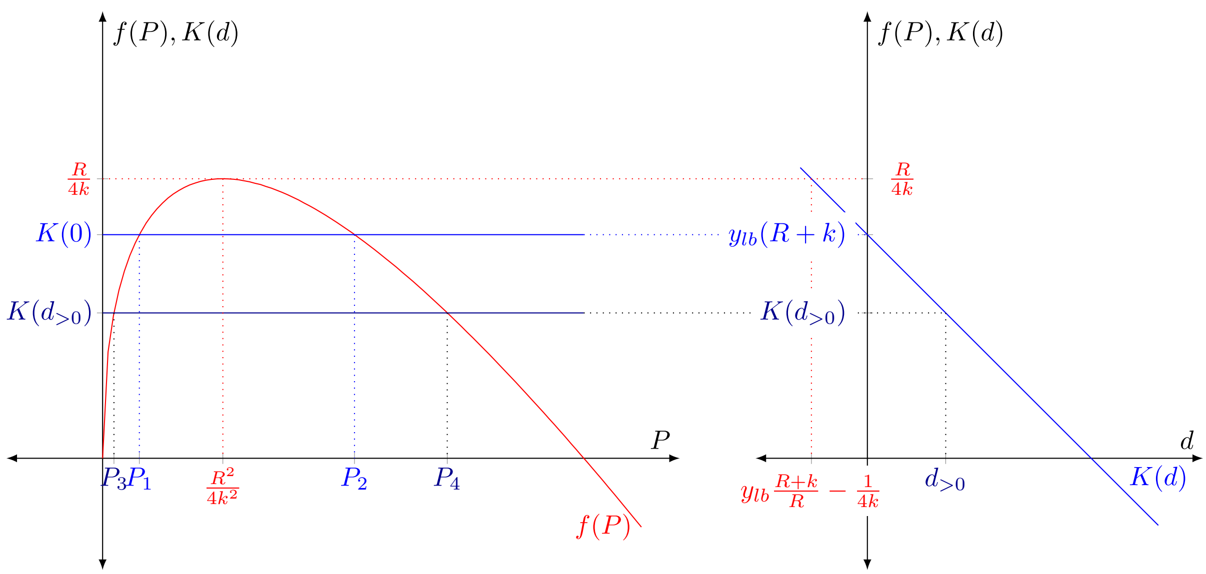
$$f(P) \equiv P^{1/2} - \kappa P y^N = y_{lb}/y^N + \kappa y_{lb} - d/y^N \equiv K(d) \tag{51}$$

[49] See Bianchi ([12]) and Biachi and Mendoza ([15]) for a discussion of Fisherian deflations and sudden stops.

Where we have set $y^T = y_{lb}$ and $(1-a)/a = Ry^N = 1$. The left hand side of (51) is a function of P and the right hand side of d. Let $f(P)$ and $K(d)$ be the former and the latter respectively. As we have assumed $Ry^N = 1$, f is increasing for $0 < P < R^2/(4\kappa^2)$ and decreasing otherwise (for $P > 0$, of course). Further, $f(R^2/(4\kappa^2)) = R/2\kappa = K(y_{lb}(1+\kappa R) - (1/2\kappa))$ and $K(0) = y_{lb}(R+\kappa)$. Figure 1 illustrates equation (51) for the described parametrization with $R^2/(4\kappa^2) \equiv P^*$.

The K locus depends on d, which is not depicted in the figure below. The f locus depends on P, that is depicted in the "x-axis". The $K(0)$ line represents the smallest possible value for d in the constrained regime (i.e. $d = 0$). Between $K(0)$ and $R/4\kappa$ the regime is not collateral constrained. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

Figure 1: Equation (51), $y^T = y_{lb}$



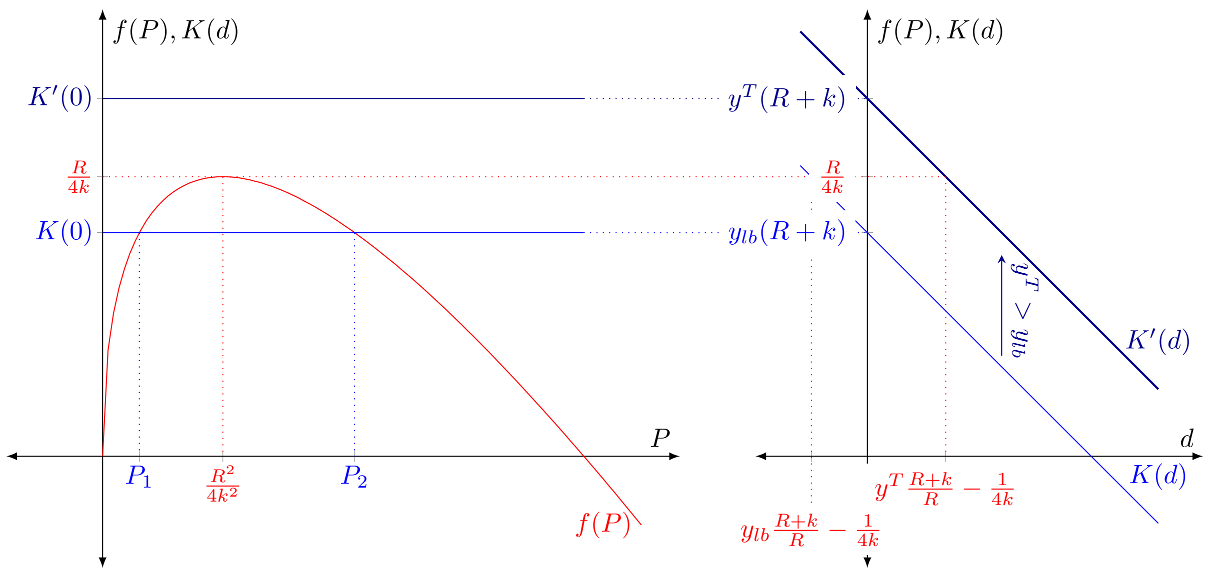
The K locus depends on d, which is not depicted in the figure above. The f locus depends on P in the "x-axis". The $K(0)$ line represents the smallest possible value for d in the constrined regime (i.e. $d = 0$). Between $K(0)$ and $R/2\kappa$ the regime is not collateral constrained. Below $K(0)$ and over the locus formed by f lie all the candidate pairs (d, P) for the constrained regime.

Note that for $d = 0$ there 2 possible exchange rate levels, $P1$ and $P2$, and a change in d with $d > 0$ can either increase or decrease P. This is depicted in points $P3$ and $P4$ in the same figure. Moreover, an increase in y^T implies that the $K(0)$ locus must jump upwards while the $f(P)$ locus remains constant as it is independent of tradable output by construction. Figure 2 illustrates this situation. Note that the collateral constraint doesn't bind when the agent saves (i.e. $d < 0$) as endowments and prices are positive. Thus, after the depicted increase in y^T, the region of possible multiple prices for a positive level of debt now includes the whole f locus.

The figures above illustrate the implications of the stochastic structure in the multiplicity of equilibrium: *as we increase the shock level from y_{lb} to y^T there is an increase in the admissible (positive) debt levels which can generate multiple equilibria.* This fact follows immediately from the definition of K.

In order to continue with the description of possible multiple equilibria we must incorporate 2 restrictions. The first comes from $P > 0$ as figures 1 and 2 don't guarantee that $P1 > 0$, even tough $P^* > 0$. From equation (51) it is clear that if $P = 0$, then $d = y_{lb}(1 + \kappa/R) \equiv d_{P=0}(y_{lb})$ and $d < d_{P=0}(y_{lb})$ implies $P > 0$. Thus, $d_{P=0}(y_{lb})$ is the upper bound on debt, d_{ub}. Further, we must insure that consumption

Figure 2: Equation (51), $y^T > y_{lb}$



An increase in y^T shifts the K line. Thus, the whole locus f contains all the candidate pairs (d, P) for the constrained regime.

remains positive. Using equation (51) again we can see that the locus, on the plane (d, P), for which consumption equals 0 is given by:

$$d = y^T + (\kappa/R)[y^T + Py^N] \tag{52}$$

Assume $\kappa/R = 1$. Clearly, in the (d, P) plane, equation (52) is a linear function $P = dR - 2y^T R$. Moreover, $P(y^T) \equiv -2y^T/y^N = -2y^T R$, where the second equality follows from $y^T R = 1$ which was imposed above, defines the intersection of this locus with the P-axis (i.e. the value for P that generates $c^T = 0$ with $d = 0$) and $d_2(y^T) \equiv 2y^T$ with the d-axis (i.e. the value for d that generates $c^T = 0$ with $P = 0$) [50]. Above this locus lie all the pairs (d, P) for which $c^T > 0$. Applying the implicit function theorem to (51) for $P > 0$, we get:

$$\frac{DP}{Dd} = -1\left(y^N[1/2P^{-1/2} - \kappa y^N]\right)^{-1} < 0 \text{ if } P < \left[2\kappa y^N\right]^{-2} \text{ and } \frac{DP}{Dd} > 0 \text{ if } P > \left[2\kappa y^N\right]^{-2}.$$

Remarkably, note at point the $P^* = \left[2\kappa y^N\right]^{-2}$, we get $DP/Dd = -\infty$. This point can be used in (51) to obtain $f(P^*) = y^{T*}(R+\kappa) = K(0)$, the value for y^T which generates P^* in (51) when $d = 0$. Any $y^T < y^{T*}$ will generate a discontinuity in equation (51) in the (d, P) plane for $d > 0$. We call this point E. Figure 3 combine figures 1 and 2 with the 2 restrictions mentioned above.

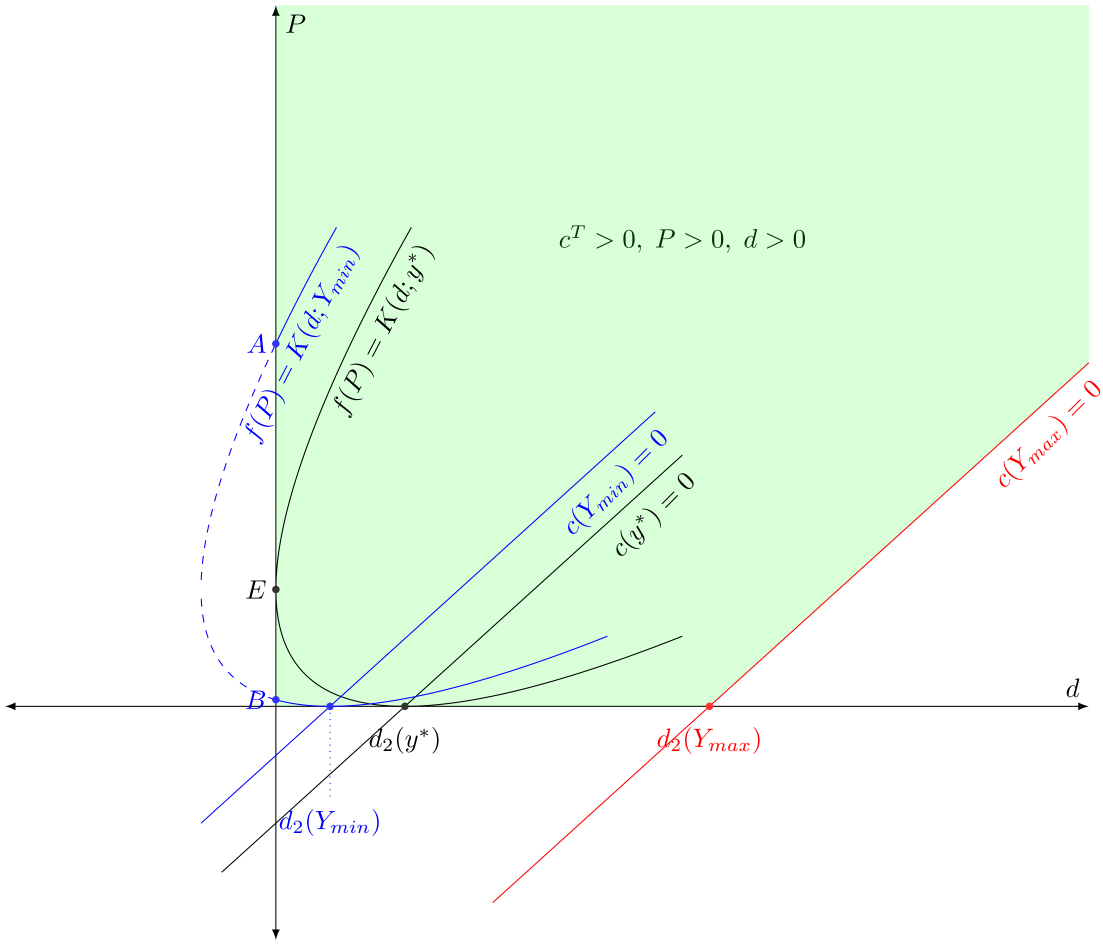
For expositional purposes we have replaced y_{lb} with $Ymin$, $y_{ub} > y^T > y_{lb}$ and $y_{ub} = Ymax$. Above the line joining the points $d_2(Y)$ and $P(Y)$, to the right of vertical axis and over the locus formed by $f(p) = K(d)$ we can find the admissible real exchange rate. If $y^T = Ymin$ and y^T is close to 0, the set

[50] In this section we are assuming the SG-U preferences. That is, the relative price P from optimality is just:

$$p(C) = \frac{1-a}{a}\left(\frac{C^T}{Y^N}\right)^{1/\xi}$$

This implies that $P = 0$ if and only if $c^T = 0$, which why the value for d that generates both restrictions is the same, d_2.

Figure 3: Genericity of Multiple Equilibria: Feasibility and Intra-temporal optimality



The feasible set of any GME is depicted in this figure. The $P - d_2$ locus defines the "non-negative consumption region". The "x-axis" defines the debt region. The "y-axis" the positive prices region. Optimality according to the intra-temporal optimization condition in the constrained regime is represented by the oval contours. Note that for low tradable income levels, this last locus is "truncated" as can be seen in figure 1. There is a level of tradable income, y^T, for which the whole f locus contains prices in the constrained regime as $K(0) = R/2\kappa$. For bigger income levels, prices are not determined in the constrained regime and, thus, they do not belong to the $f = K$ locus.

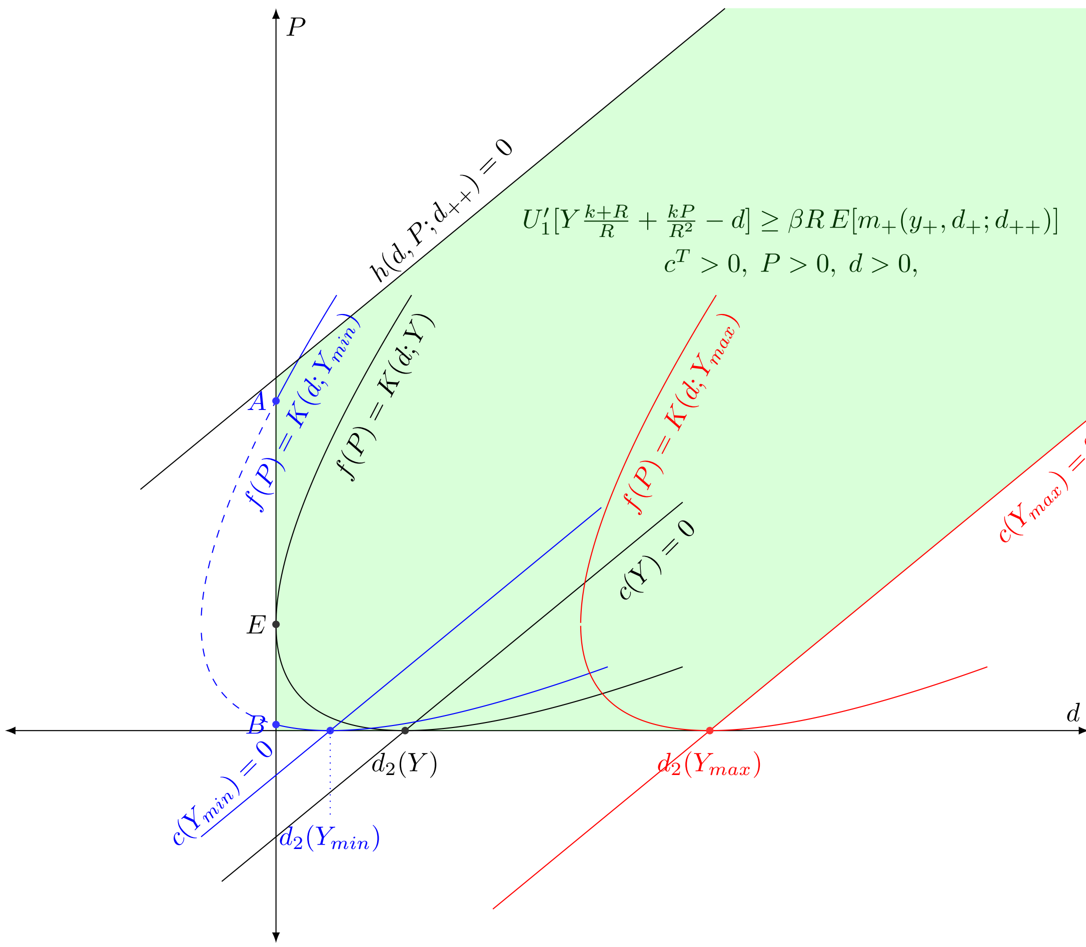
of debt levels that admit multiple equilibrium is negligible. However, when $y^T = y^{T*}$, this set is formed by all debt levels between 0 and $d_2(y^{T*})$.

Summing up, equations (51) and (52) gives the admissible pairs (d, P) in figure 3. Note that $P(y^T)$ / $d_2(y^T)$ are all functions of y^T. Thus, an increase in this variable will shift both locus to the right, as depicted in figure 3. Thus, the set of admissible equilibria increases along with y^T as can be seen by looking at the region formed by points $A - B$ on the P-axis. That is, between 0 and d_2 there are 2 admissible exchange rates for each debt level d.

The following claim states, by showing the optimality of the feasible pairs in figure 3, the existence of multiple equilibrium for a sufficiently rich set of shocks. The results are depicted in figure 4. See the supplementary material in the appendix for section 5 for a detailed discussion

Assume that the equilibrium set is compact. Then, if the set containing exogenous shocks Y has at least 3 elements, then the system of equations formed by (46) to (50) has 2 solutions for at least 2 different elements in Y (i.e. the model has multiple equilibrium).

Figure 4: Genericity of Multiple Equilibria: Feasibility and Optimality



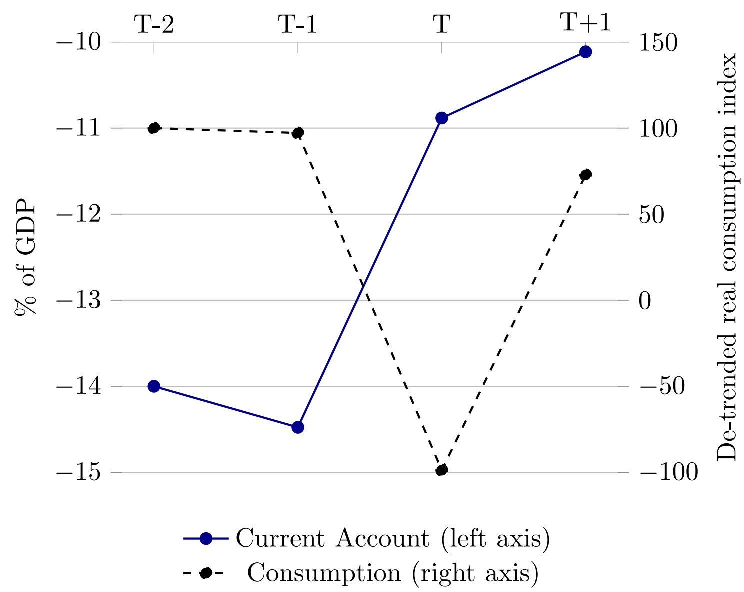
The h locus contains the (d, P) pairs which satisfy intertemporal optimality with equality in the constrained regime. Below it we can find the optimal (d, P). As h is increasing in $d_{++}(y_+)$, which represents the selection of the equilibrium correspondence Φ, we can choose h to be conveniently located. In the ergodic case, as $d_{++}(y_+)$ is restricted by Remark 3.2, the equilibrium correspondence could be a truncated version of figure 4. theorem 1 insures that the intersection of the region below h, for the ergodic selection defined in Remark 3.2, and the region defined by the oval contours in the non-negative orthant formed by the (d, P)-axis is not empty.

The last figure completely describes multiplicity. With respect to figure 3, we added the locus h. The claim above shows that: i) below h we can find the optimal pairs (d, p), ii) there are at least 2 optimal contours which insure the existence of multiple equilibrium for at least 2 shocks.

5.2 Dynamic Behavior

In this subsection we show that an ergodic GME, summarized by equations (46) to (50), is capable of generating 2 different types of sudden stops. At the same time, each of these events can be divided in 3 phases. i) The pre-sudden stop, characterized by an increase in the current account deficit. ii)The sudden stop itself, which consists of a sharp drop in consumption and a current account reversal. iii) The post-sudden stop. The events are mainly differentiated by phase iii). In the first type, exemplified by Portugal in 2008, there is a moderate recovery in consumption. In the second type, represented by Spain

Figure 5: Anatomy of a Sudden Stop: Portugal 2008



The figure is constructed using the data set in Pierri, et. al. (2018). The authors refine the definition of a sudden stop to incorporate the 3 phases mentioned in section 5.2. The dotted line represents the current account divided by GDP, both at current prices. The left index is measured in porcentaje points. The right index, in full line, contains a consumption index constructed using the de-trended (by Hodrick and Prescott, HP) series at constant prices. The base of the index is 100 at "T-2", where "T" is the date of the sudden stop (2008).

in 2008, consumption keeps falling after the sudden stop. Figures 5 and 6, borrowed from Pierri, et. al. ([59]) illustrate this situation.

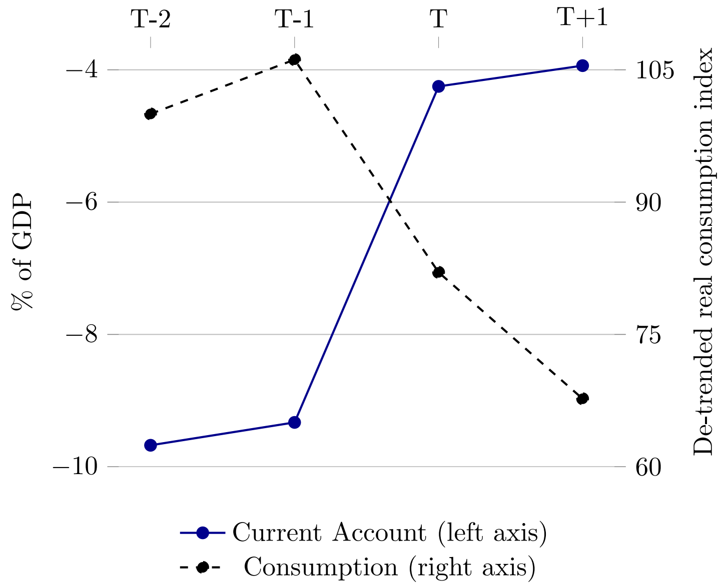
Moreover, the GME is capable of simulating a Fisherian deflation" (i.e. a path of real exchange rate depreciation coupled with falling consumption) and a sudden stop without relying on a large, unanticipated shock that impose the loss of access to capital markets by assumption. This shock is typically represented by a sudden change in a (non-stochastic) parameter or a change in the support of the distribution of exogenous shocks. Neither of these assumptions are required once we change the type of SCE we study from RE to more general GME representations. To our knowledge, this is the first attempt to show that a general equilibrium model with occasionally binding price dependent collateral constraints is capable of generating a collapse in borrowing followed by a "financial accelerator effect", caused by de-leveraging, and at the same time preserve the ergodicity of the dynamical system which, as will be shown in section 5.3, is essential for simulating and calibrating the model.

To model a sudden stop we must show the existence of an abrupt reversal in foreign borrowing. By "abrupt" we mean that it must be an intra-period event. This is the *same* effect caused by a sharp reduction in κ but without imposing an exogenous event (i.e. an event that can not be explained within the theoretical framework embodied in the model). In order to show the existence of such an event the multiplicity of equilibria will be essential.

Let "T" in figures 5 and 6 be the time of the sudden stop. Before that, in phase i), we are in the unconstrained regime. In the case of Portugal, we observe a small decrease in consumption (3%) and an increase in the current account deficit. By setting $y^T_{T-2} = y^T_{T-1} = y_{lb}$, as the unconstrained regime can be thought as a standard partial equilibrium concave savings problem, we can match the observed path [51]. In the case of Spain, we observe both an increase in consumption and a reduction in the current account deficit. The same arguments can be used to show that this path can also be generated in the

[51] The proof follows immediately from Lemma 13 and 16 in the appendix of section 4

Figure 6: Anatomy of a Sudden Stop: Spain 2008



unconstrained regime for a sequence of shocks which satisfies $y^T_{T-2} > y^T_{T-1} > y_{lb}$ [52]. Then, suppose that for some debt level d and $y^T = y_{lb}$, the collateral constraint is binding. This happens in period T without loss of generality [53]. Then, equation (50) implies:

$$U'\{A_1(y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d)\} \geq \; E(U'\{A_1(y^T_+ - \kappa\{y_{lb} + py^N\} + R^{-1}\overline{d})\}) \tag{53}$$

Where $\overline{d}$ defines the ergodic selection $\varphi \sim \Phi$ in theorem 1 and the right hand side of equation (53) follows from the existence of an envelope shown in section 3.

One of the most relevant aspects of a GME is that it is computed "backwards". That is, we are interested in finding d, instead of d_+ as in the RE It turns out that this fact can be used, combined with the primal version of the Euler equation (53), to obtain a sudden stop without requiring a jump in the exogenous parameters of the model. Let c, d_+ be the consumption and debt level tomorrow in (53). Note that, as we are in the constrained regime and P is increasing in tradable consumption, any $c_* < c$ will also satisfy (53). Take a pair (d_*, P_*) with $y_{lb} + py^N > y_{lb} + p_*y^N$, where $P(c^T) \equiv p > P(c^T_*) \equiv p_*$ with p and p_* defined by the intra-temporal optimality condition in the system defined by equations (46) to (50). Now, the discussion in the preceding section implies that there are 2 possible branches for the equilibrium selection. Taking the lower one, we know that p is decreasing in d as the branch has negative slope (see figure 4), which in turn implies $d < d_*$. Thus, $c = y_{lb} + R^{-1}\kappa\{y_{lb} + py^N\} - d > y_{lb} + R^{-1}\kappa\{y_{lb} + p_*y^N\} - d_* = c_*$ as desired. As equation (53) holds with inequality, this implies that the level of debt tomorrow is binding at p_*, that is $d_{+,*} = \kappa\{y_{lb} + p_*y^N\}$.

The discussion above suggests that it is possible to set the level of next period borrowing such that it is binding for the new level of consumption c_* as equation (53) is allowed to hold with strict inequality. Remarkably note that there exist a $\kappa' < \kappa$ such that $\kappa'\{y_{lb} + py^N\} = d'_+ = \kappa\{y_{lb} + p_*y^N\}$ which implies that $\kappa\{y_{lb} + p_*y^N\}$ *is a sudden stop level of debt* if the pair (d_*, p_*) implies a sharp contraction in consumption and a reversal in the current account. The first fact was already shown. Note that the order of magnitude of this "recession" depends on the level of consumption reached in phase i) before the

[52] See Lemma 16 in the appendix for section 4.

[53] Lemma 14 in the appendix containing the proofs of section 4 implies that starting from any initial condition, the collateral constraint binds in finite time with positive probability.

sudden stop which, given the unconstrained nature of this phase and the smoothness of consumption in that regime, is similar to c and bigger than c_*. Thus, *memory in the form of a sequence* $y^T_{T-n}, ..., y^T_{T-1}$, *matters in order to capture the quantitative properties of the sudden stop* and the GME is capable of capturing it. Technically, the compactness of the SCE insures that we can adjust the severity of the crises along with a finite lower bound on consumption, both proved in lemma 2

It remains to show that the jump from (d, p) to (d_*, p_*) generates a current account reversal. Let $d_+, d_{*,+}$ be the level of next period debt associated with c, c_* respectively. In order to generate the mentioned reversal, we must have $d_{*,+} - d_* < d_+ - d$. As $d_{*,+} = y_{lb} + p_* y^N < d_+ = y_{lb} + p y^N$ and $d < d_*$, the desired result follows. In order to connect $d_{*,+} - d_*$ with the current account level before the sudden stop, note that, by the arguments used to show phase i), a sequence of negative shocks to income, $y_{t-n} = y_{lb}$ with $n = 2, 3...N$, will generate an increasing sequence of debt. Thus, $d_{t-n} > d_{*,+}$ without loss of generality. To sum up, the sudden stop generates a *reduction in consumption and a current account reversal* as desired. Finally, as this event happens for a particular trajectory of exogenous shocks $y_0, y_1, ..., y_T$ in finite time, the event has positive probability but can be considered a *"rare event"* as noted by Mendoza and Smith ((([48])).

For the sake of concreteness, the existence of a Fisherian deflation is postponed to the appendix.

It is sometimes observed that after a sharp depreciation, it follows a recovery in consumption coupled with a real appreciation and a current account improvement. This is the case of Portugal 2008 in phase iii). A GME is capable of replicating these facts as a path in the unconstrained regime. In particular, the model associates an increase in the national income with this type of phase iii). That is, we must observe an increase in $y^T_{T+1} - d_+$. This shock implies an increase in consumption which generates the real appreciation due to the intra-temporal optimality condition $p(c^T)$.[54] Moreover, the concavity of the utility function implies $d_{++} < d_+$, as it is possible to smooth consumption in an unconstrained economy. Thus, the observed current account improvement follows.

5.3 Empirical Procedure, Algorithms and Simulations

We now turn to the quantitative implications of the results presented in sections 3 and 4. Taking into account the lack of a closed form solution, we must take care of the numerical approximation of the model presented in section 2.

We first show how to compute the ergodic selection in the GME. It can be used to simulate a recurrent, and thus ergodic, behavior as the stochastic paths visit the atom in some type of crisis. In particular, in section 4 we showed that the process hits the atom if consumption and debt are bigger than the atomic level. However, this can happen with very low probability. Numerically, given the characteristics of the model, we can find recurrent sets that are observed more often, affecting significantly the data at low frequencies. Following the theoretical results in section 4, each ergodic selection has a unique invariant measure. Thus, we have to find (numerically) another selection. Each selection is associated with d_{++} in the Euler equation. The one found in section 4 is invariant with respect to (y, d). This is convenient to prove its existence but it forced us to construct different paths of exogenous shocks for each initial condition, affecting the frequency in which this atom is observed. In order to circumvent this problem, we can connect d_{++} with (y, d). Thus, we can use "shorter" paths to hit the atom, increasing its frequency.

Using this selection, we solve the model for a parameter set borrowed from the empirical literature and compute the effects of a change in the interest rate in the long run of the model. The discussion in section 5.2 shows that it is possible to remove the ergodic component of any selection in a GME simply by visiting a different point every time the collateral constraint binds. However, this selection is still time invariant and thus, stationarity We compute the difference between simulations generated by an ergodic and a stationary GME equilibrium. We found that ergodic simulations generate smoother consumption paths or, equivalently, agents are less financially constrained. Finally, we compute a non-stationary GME. As it is expected, this equilibrium can generate large fluctuations in macroeconomic fundamentals (i.e. current account) using standard preferences and with the same shock structure. That is, simply by changing the

[54]See Lemmas 16 and 17 in the appendix for section 4.

definition of equilibrium, if we are willing to give up the long run performance of the model, the same theoretical structure is capable of generating a great range of balance of payments crises.

5.3.1 Numerical procedure

We describe how to compute an ergodic selection and use it to simulate the long (a path of length N) and short run (of length $T < N$) behavior of the model. The appendix contains additional details.

Let (d, d_+) denote the debt levels observed today and tomorrow. The presence of the collateral constraint implies that $d_+ \neq d'(d, y)$. Thus, the ergodic selection of the GME φ depends on $\varphi(d, y)$ for every $(d, y) \in K_2 \times Y$ if the collateral does not bind and on $\varphi(d_+, d, y)$ for every $(d_+, d, y) \in K_2 \times K_2 \times Y$ if it binds. The connection between the RE and the GME insured by the existence of well behaved envelopes, implies that there is a stationary structure for the GME given by equations (46)-(50) and $d'(.,.)$, the policy function in the unconstrained regime for the RE This fact implies that the GME is computationally efficient. That is, once $d'(.,.)$ is available, it can be computed fast.

The algorithm in the appendix, called GME ergodic algorithm, generates a sequence $\{p_t, d_{t+1}\}_{t=0}^{T}$ which *depends* on $p_{T+1}(y_{T+1})$ and $d_{T+1}(y_T)$ for a given point in the set Θ. This variables are pinned down by picking an ergodic selection for the GME. The results in section 4 implies that we need to constraint the paths generated by the GME using the policy function of the RE. However, this is not necessarily the case if we only need a stationary equilibrium. That is, in this last case, we may allow $d_+ \neq d'(d, y)$ even if the collateral does not bind. Thus, we are adding memory to the selection as d_+ may not depend on (d, y). Going backwards we only have 1 additional restriction to satisfy: in order to compute d_-, given d, we know that $E(m)$ includes $\varphi(d, d_-, y')$ for any $y' \in Y$. This fact insures that we can still find d_- to satisfy the restrictions associated with intertemporal optimality for any $d_- \in K_2$. Contrarily, any sequence generated from the minimal state space algorithm depends *only* on the point in Θ and the draw from (Y, q) for a given d_0, y_0. That is, $\varphi \in \Phi$ *does not necessarily satisfied* $d_{T+2}(y_{T+1}) = d'(d'(d_T, y_T), y_{T+1})$ as it is the case for the minimal state space algorithm. Taking into account the results in section 5.2, this distinction is critical as it allows to choose d_+ to be the collateral constrained level and to vary d_- along with the degrees of freedom in $E(m)$ to generate a "crises".

Finally, the function which maps $[y_t, p_t, d_t] \longmapsto [y_{t+1}, p_{t+1}, d_{t+1}]$ for each $y_{t+1} \in Y$, which defines the Markov kernel in the GME, *does not necessarily satisfy* the structural properties required to prove the existence of a RE Thus, the path $\{p_t, d_{t+1}\}_{t=0}^{T}$ is more flexible to capture the different phases of a macro crises as the transition function in the GME can be computed pointwise as in the SCE. This is the *numerical implication of expanding the memory in the recursive equilibrium notion as the transition function is computed exactly as in the SCE* (i.e. pointwise for each element in the draw from (Y, q)).

The process is shown to be stationary by making the length of the simulation arbitrarily large. However, if we iterate forward, using d_T, d_{T+1} and computing $d_{T+2}(y_{T+1})$ for every $y_{T+1} \in Y$ we are obtaining a path from the SCE which is not necessarily stationary. This path may be interpreted that as a prediction of the model that is not constrained by the structure imposed by ergodicity or stationarity.

The appendix also describes a non-stationary GME Algorithm.

In this case the sequence $\{p_t, d_{t+1}\}_{t=0}^{T}$ *depends* on the histories of the form $p_y(y^t)$, $d_t(y^{t-1})$ with $y^t = y_0, ..., y_t$. Thus there is a trade off: *we gain flexibility with respect to the stationary / ergodic GME in order to incorporate more "memory" from the SCE but in return we can not claim that these paths are connected with the steady state of the model.* Thus, depending on the relevance that we assign to the long run behavior, we can choose between the GME ergodic/ stationary or non-stationary procedure.

5.3.2 Results

We now solve the model. The table below contains the parameters, borrowed from Pierri, et. al. ([59]).

Table 1: Parameters

Parameter	κ	β	σ	ξ	a	z_l	z_h	$p(z_l)$	R	R'
Value	0.3	0.99	2.0	0.5	0.5	0.5	1.5	0.2	1.05	1.025

Table 2: Summary of Simulations Statistics for Consumption

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	1.0497	0.450	1.0501	0.453
Stationary	1.01	0.52	1.04	0.53

We now show the results of simulating the ergodic and the stationary process. We present the effects of a reduction in the interest rates in both cases for consumption (Table 2) and debt (Table 3).

Table 3: Summary of Simulations Statistics for Debt

Statistics	Mean(R)	STD(R)	Mean(R')	STD(R')
Ergodic	0.230	0.086	0.235	0.094
Stationary	0.27	0.06	0.29	0.07

The statistics are reported at distinct truncation levels as there are some cases for which 2 decimal positions are not enough to differentiate between simulations. A reduction in the interest rate generates the expected changes in ergodic and stationary simulations: an increase in consumption and thus a reduction in the savings rate which implies more debt. However, there are at least 2 connected differences : stationary simulations overestimate i) the elasticity of average consumption and debt with respect to the interest rate, ii) the volatility of consumption for the same interest rate, which implies that debt is less volatile. For the first fact, the intuition goes as follows: as the atom is not directly affected by the change in the interest rate, only through its effect on the unconstrained policy function for the RE, and ergodic simulations are generated as a sequence of recurrent sets which has a regeneration point in the atom, average observed endogenous variables are not severely affected. For the second fact, note that in the ergodic simulation debt is regenerated to a very low level as by construction the atom is defined to hit the collateral constraint with equality. Thus, de-leveraging is more significant in an ergodic crises, which implies that the economy has more time to accumulate debt and to smooth consumption.

The take away point from the above results is related to the invariance of the atom with respect to parameter changes. The most direct way to change the ergodic distribution is to affect the regeneration point. In this case, numerically, the atom does not change significantly after the interest rate shock even tough the unconstrained policy function and the value of the collateral are both affected. In particular, the atom is given, as described in the appendix, by:

$$d'(d_*, y_{lb}; R) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb}; R)/R) - d_*)} \right) \right]$$

After the change in the interest rate both the left and right hand side of the equation above rises as $d'(d_*, y_{lb}; R)$ and $d'(d_*, y_{lb}; R)/R$ goes up. Thus, the change in d_* is not significant. Finally, we present a non-stationary simulation in the table below.

Table 4 reflects the flexibility contained in the model. By changing selections, the SCE is able to replicate a sharp reversion in the current account, as it is frequently observed in data, without requiring a change in the parameters (i.e. a reduction of κ or y_{lb}).

Table 4: Non Stationary Crises

Non Stationary Simulations	$t-1$	t	$t+1$
Current Account / GDP	-10%	-8%	$+5\%$

6. Conclusions and Directions for Future Research

In this paper, we have proposed a new set of tools for characterizing models of financial crises in small open economies in the presence of multiple dynamic equilibria. We have proven the first results in this literature in the existence of both SCE and RE, and provided a complete qualitative theory of the set of RE. In principle per the set of RE, our methods are constructive, and provide a global basis for the existence of robust multiplicities of RE.

We then show our GME approach to stochastic equilibrium dynamics can replicate the short run dynamic of a crises and still be connected with long run stylized facts. These properties depend on the existence of a sequential competitive equilibria and minimal stat
space Markov equilibria. Once these elements are available, we can classical results in stochastic process the existence of ergodic GME selectors, we can compare the stochastic properties of RE to based upon ergodic selectors versus GME selectors associated with invariant measures, we can study nonstationary GME selectors, and we can compare the stochastic properties of Sudden Stops generated by these different GME selectors versus selecting exclusively from the set of RE. Our methods are also easily implemented numerical, as GME can be efficiently computed using a standard laptop.

The results in this paper are derived for a restrictive set of assumptions to illustrate the power of our approach (i.e., the case where the exogenous endowment shocks must be drawn from a finite set, are iid, and we assume $\beta R < 1$. For the set of SCE, we do impose some conditions on asymptotic bounds for marginal utility (to obtain needed compacted for proving the existence of SCE), which in turn in principle could affect the homotheticity of equilibrium paths. We are able to relax the sufficient condition on preferences to the standard case in the literature relative to the set of RE, but many of our results in this paper how for me general environments of preferences, shocks, and cases where $\beta R \geq 0$. (e. g., see [60]).

The results obtain for the minimal state space RE are of independent interest, taking into account the relevance of the high / low borrowing for the design of economic policies. Under rational expectations, this equilibrium can be self-fulfilling. However, it remains to be shown if this type of equilibrium can co-exist under another rationality assumption such as learning, among others.

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Appendix

Proofs for section 2

We now turn to the characterization of the SCE. In order to get the paper self-contained, we write the primal version of the first order conditions which will be useful to prove the long run properties of the equilibrium. We keep the notation $c_1 \equiv c^T, c_2 \equiv c^N$ in line with Assumption 1.
For any given sequence of prices $p = \{p_t\}_{t=0}^{\infty}$, from the first order conditions in (7-10), we can obtain the following two expressions critical in characterizing the stochastic structure of binding collateral constraints:

$$[c_2(y^t)][\{\beta^t U'(A(c(y^t))\}\{-A_1(c_1(y^t))p(y^t) + A_2(c_2(y^t))\}] = 0, \quad y^t - a.e \tag{54}$$

$$\begin{aligned}[\kappa\{y^t + p(y^t)y^N\} - d(y^t)][U'(A(c(y^t))A_1(c_1(y^t)) \\ -E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))] = 0, \quad y^t - a.e\end{aligned} \tag{55}$$

where E_t can be obtained using the $y_t th$ row of the transition matrix if the cardinality of this shock set Y is finite, or by integrating using the density associated with χ evaluated at y_t if Y is an uncountable set. In the i.i.d case, we have $E_t = E$.
We make a few remarks on (54) and (55). First, the characterization of optimal solutions uses the primal formulation of the problem, and hence is written in terms of the "complementary slackness" version of the Karush-Kuhn-Tucker (KKT) equations. In particular, in either equation (54) or (55), the first bracket contains the inequality constraint, $c_2 \geq 0$ in equation (54) or $\kappa\{y^t + p(y^t)y^N\} - d(y^t) \geq 0$ in equation (55), and the second bracket consists of the derivative of the objective function with respect to the control, $c_2(y^t)$ in equation (54) and $d(y^t)$ in (55). Additionally, note we have eliminated the control, c_1, and the restriction in equation (2) from the KKT system. Formally, $c_1(y^t)$ must be replaced with $-p(y^t)c_2(y^t) - d(y^{t-1}) + y^t + p(y^t)y^N + \frac{d(y^t)}{R}$. This last issue is relevant for the dual formulation of the problem as it allows us to avoid dealing with the Lagrange multiplier associated with (2). It turns out that the dual representation is more difficult to characterize in terms of its dynamic behavior when compared with the primal version.
Now, in order to close the characterization of the model, we need a terminal condition on the right hand side of the Euler equation $\beta^t E_t(U'(A(c(y^t y_{t+1}))A_1(c_1(y^t y_{t+1}))$, which results after iterating equation (55) (see Constantinides and Duffie ([22]) for a discussion). Under assumptions 1-a.i, 1-a.iii, 1-e, and 1-f, this requirement will be satisfied. The relevance of these assumptions, their relationship with the restriction we place on βR and the sufficient conditions for the compactness needed to obtain the existence of SCE will be proved in lemma 1, stated in the body of the paper. Note that the results below are necessary conditions. The associated sufficient conditions for existence will be proved in theorem 1.

Proof of Lemma 2

Proof. I) Let $Y = [y_{min}, y_{max}]$. From assumption 1-g), lemma 1 in Braido ([18]) implies that $-d(y^t) < \frac{R}{1-\rho} \equiv k_{2,min}$ uniformly in $y^t \in \Omega$ with ρ sufficiently close to 1. From assumption 1-a3), 1-b), 1-c) and 1-d), equation (54) implies $p(y^t) = \frac{A_2(c_2(y^t))}{A_1(c_1(y^t))}$. Definition 1-2), equation (54), assumption 1-e), 1-f) then imply $p(y^t) \in [\frac{y^N}{cu_1}, \frac{y^N}{cl_1}] \equiv K_3 \equiv [k_{3,min}, k_{3,max}] \quad y^t - a.e.$. Then, the collateral constraint implies $d(y^t) \leq \kappa(y_{max} + k_{3,max}y^N) \equiv k_{2,max}$. Then, $d(y^t) \in [k_{2,min}, k_{2,max}] \equiv K_2 \quad y^t - a.e.$. Finally, using K_2 and K_3, K_1 can be derived using equation (2) and Definition 1-2). Using these results, it is straightforward to verify that $lim_{t\longrightarrow\infty}\ \beta^t E_t(U'(A(c_t^*))A_1(c_t^*)) = 0$. Note that the integral in E_t is taken with respect to either the density associated with $\chi(y_t)$ or the $y_t jth$-row in the transition matrix in case Y has finite cardinality. Thus, this result also holds $y^t - a.e.$. Then, the conditions in Lemma 1 in Kubler and Schmedders ([38]) hold, which implies that equations (54) and (55) together with the terminal condition on $\beta^t E_t(U'(A(c_{t+1}))A_1(c_1(t+1)))$ are equivalent to definition 1, as desired. ■

Proof of Theorem 3

Proof. As in Kubler and Schmedders ([39]), we will start with a truncated economy $t = 0, ..., T$ and then extend the argument by induction. In order to show the theorem, we need to rewrite the conditions in definition 1. Any SCE satisfies conditions A and B: Condition A

$$Max_{d(y^t)} \sum_t \sum_{y^t} U(c_1(y^t), y^N)\mu(y^t)$$

$$s.t$$

$$c_1(y^t) = y_t^T - d_t + min\left\{d_{t+1}/R, R^{(-1)}\kappa(y_t^T + p(y^t)y^N)\right\}$$

and Condition B

$$p(y^t) = \frac{A_2(y^N)}{A_1(c_1(y^t))}$$

Because of the linearity of the restriction, we can substitute c_1 into the objective function in condition A. Thus, because of lemma 1, the maximization problem is only restricted by the fact that $d(y^t) \in K_2$ for all y^t. Thus, $\{d_{t+1}(y^t)\}_{y^t} \in K_2^{(Y)^T}$, where $(Y)^T$ is the number of possible nodes. Note that given p, this is a strongly concave problem restricted by a continuous correspondence. By Berge's maximum theorem, we know that $d_{t+1}(d_0, y^t, p)$ will be a continuous function of $p \in K_3^{(Y)^T}$ for all $d_0 \in K_2$ and $y^t \in (Y)^T$. ∎

Supplementary Material for section 2.2

This section contains concrete examples of utility function which ensure that marginal utility is bounded.

$$Mod. \quad CD \quad (c_1 + \gamma)^\alpha (c_2 + \gamma_1)^\beta$$
$$u : X \to \Re, \quad X \supseteq \Re^2_+, \quad \gamma, \gamma_1 > 0, \quad c \in X \Rightarrow c + [\gamma, \gamma_1] > 0$$

The "Mod. CD" preferences are defined over a consumption set which includes the "zero"-vector and γ insures that marginal utility remains bounded above over the entire consumption set, X. The "Mod. LOG" and "Mod. CES" are similar. Just replace $(c_1 + \gamma)^\alpha \ (c_2 + \gamma_1)^\beta$ by $ln\,(c_1 + \gamma) + ln\,(c_2 + \gamma_1)$ and by $(a\,(c_1 + \gamma)^\alpha + a\,(c_2 + \gamma_1)^\alpha)^{(1/a)}$ respectively with $a > 0$. The "Mod. CES 2" are rather different as they are intended to keep MU bounded away from zero. In particular,

$$Mod. \quad CES\ 2 \quad (a_1\,(c_1)^{(1-\alpha)} + (1-\alpha)a_2c_1 + b_1\,(c_2)^{(1-\alpha)} + (1-\alpha)b_2c_2)^{(1/(1-\alpha))}$$
$$u : X \to \Re, \quad X \supseteq \Re^2_+, \quad \alpha > 1, \quad a_1, a_2, b_1, b_2 > 0$$

If we combine "Mod. CES" with "Mod. CES 2" we insure that MU remains bounded above and away from zero, which in turn guarantees that p is positive and finite. These are sufficient conditions for the existence of an a.e. compact equilibrium.

Proofs for Section 3

Proof of Lemma 6

Proof. The proof of part takes place in five steps. We first show the operator $A(c; C^T(S))(d,y)$ as for each $C^T(S) \in \mathbf{C}^f$, $A(c; C^T(S))(d,y) : \mathbf{C}^p \to \mathbf{C}^{p*}$, where $\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; C^T) \subset \mathbf{C}^p$, and

$$\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; C^T) = \{c \in \mathbf{C}^p | \; c = \inf\{\hat{c}(d,y), C_c^T(D, Y, C_i^T(S)), \hat{c} \in \mathbf{C}^p, C^T(S) \in \mathbf{C}^f\} \tag{56}$$

is closed $\mathbf{C}^p$. Second, we show the mapping $A(c, C^T(S))(d,y)$ is jointly monotone on $\mathbf{C}^p \times \mathbf{C}^f$, and order continuous in $c \in \mathbf{C}^p$ for each $C^T \in \mathbf{C}^f$. Third, we show the greatest fixed point of $A(c; C^T(S))(d,y)$ (denoted for now by $c^*(C^T(S))(d,y)$) is strictly positive, can be computed by successive approximations from an initial $c_0 = c_{\max}$ for each $C^T \in \mathbf{C}^f$. The fourth step, we show the greatest fixed point is increasing in $C^T(S)$ on $\mathbf{C}^f$. Finally, in the fifth step, we show $c^*(C^T(S), \beta, R, \kappa)(d,y)$ is the *unique* strictly positive fixed point in $\mathbf{C}^p$ of $A(c; C^T(S))(d,y)$ for each $C^T \in C^f$.[55]

Step 1: $A(c; C^T(S))(d,y) : \mathbf{C}^p \to \mathbf{C}^{p*}$. Fix $C^T \in \mathbf{C}^f$, $c \in \mathbf{C}^p$, and $s = (d,y,S)$. First, the operator $A(c; C^T(S))(d,y)$ is well-defined. So see this, observe when $c(d,y) \in \mathbf{C}^p$, $c(d,y) = 0$ for any state (d,y), $C(d', y'; c, C^T) = 0$, we define $x^*_{uc}(s^e, c, C^T) = A(c; C^T(S))(d,y) = 0$. So consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $(c, C^T) \in \mathbf{C}^p \times \mathbf{C}^f$, the mapping $Z^*_{uc}(x, s^e; c, C^T)$ in equation (34) is strictly decreasing and continuous in x, for any $(d,y,S; c, C^T)$. Compute an implicit mapping $x^*_{uc}(d,y,S; c, C^T)$ in the following equation:

$$Z^*_{uc}(x^*_{uc}(d,y,S;c,C^T), s; c, C^T) = 0$$

If $x^*_{uc}(d,y,S;c,C^T)$ exists, as Z^*_{uc} is strictly decreasing and continuous in x under Assumption 1, $x^*_{uc}(d,y,S;c,C^T)$ will necessarily be unique (hence, a function). When $x \to 0$, $Z^*_{uc}(x, s^e, c, C^T) \to \infty$ by the Inada condition in Assumption 2. Further, as x gets sufficiently large, $C((R(x - y^T + d), y', R(x - y^T + d, y') \to 0$, hence $Z^*_{uc} \to -\infty$. Then, by the intermediate value theorem, $x^*_{uc}(d,y,S,c,C^T)$ exists (hence, it is well-defined as a function).

Next, we show $x^*_{uc}(d,y,S;c,C^T) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$. We first show $A_{uc}(c; C^T(S))(d,y) = x^*_{uc}(d,y,S;c,C^T) \in \mathbf{C}^p$. Again, when $c(d,y) \in \mathbf{C}^p$, $c(d,y) = 0$ in any state (d,y), $\Rightarrow C(c, C^T) = 0$; hence, $x^*_{uc}(d,y,S;c,C^T) = 0 \in \mathbf{C}^p$. Therefore, consider the case when $c \in \mathbf{C}^p$, $c(s^e) > 0$. As $C^T \in C^f$, for fixed $c \in \mathbf{C}^p$, Z^*_{uc} in (34) is (strictly) decreasing in d, (strictly) increasing in y, and strictly decreasing in x; hence, at such $s = (d,y,S)$, the root $x^*_{uc}(d,y,S;c,C^T)$ is decreasing in d, and increasing in y. Further, when $d_2 \geq d_1$ and $y_1 \geq y_2$, by the concavity of utility in Assumption 1, we have from the definition of the $x^*_{uc}(d,y,S,c,C^T)$ in Z^*_{uc} the following inequality

$$\frac{U_1(x^*_{uc}(d_1,y_1,S;c,C^T), y^N)}{R} \leq \int \beta U_1(C(R(x^*_{uc}(d_2,y_2,S;c,C^T) - y_2^T + d_2), y', R(x^*_{uc}(d_2,y_2,S;c,C^T) - y_2^T + d_2), y')\chi(dy')$$

hence, for the root $x^*_{uc}(d,y,S;c,C^T)$ must make the right side of the above expression fall at $x^*_{uc}(d_2,y_2,S;c,C^T)$ in a new solution, which implies:

$$x^*_{uc}(d_1,y_1,S;c,C^{T*}) - y_1^T + d_1 \geq x^*_{uc}(d_2,y_2,S;c,C^{T*}) - y_2^T + d_2$$

or

$$y_1^T - d_1 - x^*_{uc}(d_1,y_1,S;c,C^{T*}) \leq y_2^T - d_2 - x^*_{uc}(d_2,y_2,S;c,C^T)$$

Therefore, for each $C^T \in C^*$, $A_{uc}(c; C^T(S))(d,y) = x^*_{uc}(d,y,S;c,C^T) \in C^p$.

Finally, as $A_{uc}(c; C^T)(s^e) \in C^p$, and $A_c(C^T)(S)$ is independent of (d,y) at $C^T(S) \in C^f$, $A(c; C^T)(s^e)$ $= \inf\{A_{uc}(c; C^T)(s^e), A_c(C^T)(S)\} \in \mathbf{C}^{p*}$.Therefore, we conclude $A(c; C^T(S))(d,y) : \mathbf{C}^p \to \mathbf{C}^{p*}$.

[55]This last step will also imply that the unique strictly positive fixed point is continous in the topology of pointwise convergence in C^T. As it will also be isotone on C^T, the resulting second step operator will be *order continuous* on C^T. That mean in main theorem of this section, *all* our arguments can be make constructive as mentioned in section 3. See Pierri and Reffett ([60]) for a discussion.

Step 2: *$A(c, C^T(S))(d, y)$ is monotone (increasing) on $C^p \times C^f$.* Take $x_1 = (c_1, C_1^T)$ and $x_2 = (c_2, C_2^T) \in \mathbf{C}^p \times \mathbf{C}^f$, with $x_1 \leq x_2$ under the pointwise partial order on the product space $\mathbf{C}^p \times \mathbf{C}^f$. First, consider the case $0 \leq x_1 \leq x_2$, where in some state (d, y, S), either $0 = c_1(d, y)$ or $0 = C_1^T(S)$, Then, by definition of the operator $A(c, C^T(S))(d, y)$, $A(c_1, C_1^T)(d, y, S) = 0 \leq A(c_2, C_2^T)(d, y, S)$. So, now consider the case where $0 < x_1(d, y) \leq x_2(d, y)$, so in all states, $0 < c_1(d, y)$ and $0 < C_1^T(S)$. Then, we have from the definition of x_{uc}^* in Z_{uc}^* the following inequality:

$$\begin{aligned}
&\frac{U_1(x_{uc}^*(d, y, S; c_1, C_1^T), y^N)}{R} = \\
&\quad \int \beta U_1(C_1(R(x_{uc}^*(d, y, S; c_1, C_1^T) - y^T + d), y', R(x_{uc}^*(d, y, S; c_1, C_1^T) - y_2^T + d_2), y')\chi(dy') \\
&\quad \geq \int \beta U_1(C_2(R(x_{uc}^*(d, y, S; c_1, C_1^T) - y^T + d), y', R(x_{uc}^*(d, y, S; c_1, C_1^T) - y_2^T + d_2), y')\chi(dy')
\end{aligned}$$

where for $i = 1, 2$, the subscript on continuation consumption is used to denote.

$$C_i(c, C^T)(d, y, S) = \inf\{c_i(d, y), C_c^T(D, Y, C_i^T(S))\}$$

where recall $C_c^T = (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(D, Y)Y^N$. Therefore, as Z_e^* is strictly falling in x, we have

$$x_{uc}^*(d, y, S; c_1, C_1^T) \leq x_{uc}^*(d, y, S; c_2, C_2^T)$$

Then, if the implied debt at $d_{x_{uc}}(d, y, S; c_1, C_1^T) \leq \kappa(Y^T - D + p(C_1^T)Y^N$, then

$$\begin{aligned}
A(c_1, C_1^T(S))(d, y) &= A_{uc}(c_1, C_1^T(S))(d, y) \\
&= x_{uc}^*(d, y, S; c_1, C_1^T) \\
&\leq x_{uc}^*(d, y, S; c_2, C_2^T)
\end{aligned}$$

else,

$$\begin{aligned}
A(c_1, C_1^T(S))(d, y) &= (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C_1^T(S))Y^N \\
&\leq (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C_2^T(S))Y^N
\end{aligned}$$

In either case, for $A(c_2, C_2^T(S))(d, y)$, we have $A(c_1, C_1^T(S))(d, y) \leq A(c_2, C_2^T(S))(d, y)$. So the operator $A(c, C^T(S))(d, y)$ is monotone.

Next, we prove $A(c; C^T(S))(d, y) : \mathbf{C}^p \to \mathbf{C}^{p*}$ is order continuous for each fixed $C^T(S) \in \mathbf{C}^f$. First, some definitions. Let X be a countably chain complete partially ordered set,[56] and $X_c = (x_n)_{n \in \mathbf{N}} \subset X$, $x_n \in X$, be a countable chain. We say a operator $A : X \to X$ for is *order continuous* if for any countable chain $X_c \subset X$, $A(x)$ (a) sup-preserving: $A(\vee X_c) = \vee A(X_c)$ and (b) inf-preserving: $A(\wedge X_c) = \wedge A(X_c)$. We remark, order continuous operators are necessarily isotone (e. g., Dugundji and Granas ([**?**], p. 15)). We now show for each $C^T(S) \in \mathbf{C}^f$, $A(c; C^T(S))(d, y)$ preserves sup operations; a similar argument works for preserving inf operations. Fix the state (d, y), and $C^T(S) \in \mathbf{C}^f$, and denote by $C_c = (c_n(d, y))_{n \in \mathbf{N}}$, $c_n(d, y) \in \mathbf{C}^p$ any countable chain in $\mathbf{C}^p$. Define $\vee C_c(d, y) \in \mathbf{C}^p$ and $\vee A(C_c; C^T(S))(d, y) \in \mathbf{C}^{p*}$, which both exist in $\mathbf{C}^p$ (resp, $\mathbf{C}^{p*}$) are both complete lattices (hence, countably chain complete). If in any state (d, y), $\vee C_c(d, y, S) = 0$, then $\vee A(C_c; C^T(S))(d, y) = A(\vee C_c; C^T(S)) = 0$. Therefore, assume for every state (d, y, S), $\vee C_c(d, y) > 0$. Then, we have the following inequalities for continuation tradeables consumption $C(c_n; C^T) = \inf\{c_n(d, y), C_c^T(D, Y, C^T(S))\}$

$$\begin{aligned}
C(\vee C_c) &= C(\vee c^n; C^T) \\
&= \inf\{\vee c_n(d, y), (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N\}) \\
&= \vee \inf_n(\{c_n(d, y), (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))Y^N\}) \\
&= \vee C(c^n; C^T) = \vee C(C_c; C^T(S))
\end{aligned}$$

[56] Let X be a partially ordered set. We say X is countably chain complete if for all countable subset X_c that are a chain (i.e., for no two elements $x_1, x_2 \in X_c$, x_1 and x_2 are ordered), $\vee X_c \in X$ and $\wedge X_c \in X$.

where in the second line $\vee c_n(d,y)$ is computed, and then the infimum over two continuous functions $(\vee c_n(d,y), (1+\frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N)$ is taken over a compact set $(d,y,S) \in \mathbf{D}\times\mathbf{Y}\times\mathbf{S}$, and hence continuous by Berge's theorem, in the third line, $\inf_n$ is computed pointwise over $(d,y) \in D\times Y$ (a compact set and hence continuous) at each $n \in \mathbf{N}$, and this collection is then increasing pointwise in n as C_c is a countable chain) and the sup is taken over $n \in \mathbf{N}$. Then, the remaining equalities follow from p continuous, and the fact that sup and inf operations over two continuous functions are each continuous over the compact set $(d,y,S) \in \mathbf{D}\times\mathbf{Y}\times\mathbf{S}$ by Berge's maximum theorem.
Using these facts, and substituting into the definition of $Z^*_{uc}(x,d,y,S;c,C^T)$, we have for the root $x^*_{uc}(d,y,S,c,C^T)$ the following equalities:

$$\begin{aligned} Z^*_{uc}(x^*_{uc}(d,y,S;\vee c_n,C^T),d,y,S;\vee c_n,C^T) &= \vee Z^*_{uc}(x^*_{uc}(d,y,S;\vee c_n,C^T),d,y,S;c_n,C^T) \\ &= \vee Z^*_{uc}(x^*_{uc}(d,y,S;c_n,C^T),d,y,S;c_n,C^T) \\ &= Z^*_{uc}(\vee x^*_{uc}(d,y,S;c_n,C^T),d,y,S;c_n,C^T) \end{aligned}$$

where the first equality follows from $U_1(c,y^N)$ continuous and $C(\vee c_n,C^T) = \vee C(c_n,C^T)$, the second line follows from Z^*_{uc} continuous (pointwise) in (x,c_n) for fixed C^T, the third line follows from Z^*_{uc} continuous in x. Then, noting that for any state where collateral constraints do not bind, we have $x^*_{uc}(d,y,S,c_n,C^T) \le A_c(C^T)(S)$, our operator $A(c,C^T(S))(d,y)$ is for each $n \in \mathbf{N}$ defined as:

$$A(c_n,C^T(S))(d,y) = \inf_n\{x^*_{uc}(d,y,S;c_n,C^T), A_c(C^T)(S)\}$$

so we have the following:

$$\begin{aligned} A(\vee c_n;C^T(S))(d,y) &= \inf\{x^*_{uc}(d,y,S;\vee c_n,C^T), A_c(C^T)(S)\} \\ &= \inf\{\vee x^*_{uc}(d,y,S;c_n,C^T), A_c(C^T)(S)\} \\ &= \vee \inf_n\{x^*_{uc}(d,y,S;c_n,C^T), A_c(C^T)(S)\} \\ &= \vee A(c^n,C^T(S))(d,y) \\ &= \vee A(c_n;C^T(S))(d,y) \end{aligned}$$

where the second equality follows again from $U_1(c,y^N)$ is continuous and $C(\vee c_n;C^T) = \vee C(c_n;C^T)$ for each C^T,and for the third equality again uses the fact that $\inf_n$ here is an increasing pointwise in n, and the sup is then taken over $n \in \mathbf{N}$. Hence, $A(c;C^T(S))(d,y)$ is order continuous in $\mathbf{C}^p$ for each fixed $C^T \in \mathbf{C}^f$, which completes the proof of Step 2.
Remark: Before proceeding to step 3, we mention that as equilibrium fixed point comparative statics be an important question in Steps 4 and 5, for the remaining steps of the proof of this lemma, we add to the notation for our operator for the parameters of interest, and remark that the operator $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is increasing in κ, and decreasing in (β,R) for fixed (c,C^T,d,y,S). To see this, noting $c \in \mathbf{C}^p$ is decreasing in d, U_1 is decreasing in c under assumption 1, Z^*_{uc} in (34) is decreasing in (R,β). So, the root $x^*_{uc}(d,y,S,c,C^T;\beta,R,\kappa)$ is decreasing in (β,R). Further, $A_c(C^T;R,\kappa)$ is decreasing in R. As our operator is defined as the infimum of two decreasing mappings, $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is decreasing in (β,R). As Z^*_{uc} is independent of κ, but $A_c(C^T;R,\kappa)$ is increasing in κ, $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is increasing in κ.
Step 3. *Existence and computation he greatest fixed point of* $A(c;C^T(S),\beta,R,\kappa)(d,y) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$. Fix $C^T \in \mathbf{C}^f$, and denote by $\Psi_A(C^T(S),\beta,R,\kappa)(d,y) \subset \mathbf{C}^{p*}$ the set of fixed points of mapping of $A(c;C^T(S),\beta,R,\kappa)(d,y) \in \mathbf{C}^{p*} \subset \mathbf{C}^p$.[57] By definition, the least fixed point is trivial, and is $c^* = 0$ for all $y \in Y$. By step 2 above, $A(c;C^T(S),\beta,R,\kappa)(d,y)$ is order continuous on $\mathbf{C}^p$. Further, $A(c^{\max};C^T(S))(d,y) \le c_{\max}$ (with strict inequality for some states (d,y)). Hence, by the Tarski-Kantorovich theorem (e.g, Dugundji and Granas ([28], p.15), the greatest fixed point $c^*(C^T(S))(d,y)$ can be computed as:

$$\wedge A^n(c^{\max};C^T(S),\beta,R,\kappa)(d,y) \to c^*(C^T(S),\beta,R,\kappa)(d,y) > 0$$

[57]Note, as as $\mathbf{C}^{p*}$ as $\mathbf{C}^{p*}$ is a subcomplete sublattce in $\mathbf{C}^p$, $\mathbf{C}^p$ is a complete lattice, this implies the fixed point of the mapping $A(c;C^T(S),\beta,R,\kappa)(d,y) \in \mathbf{C}^{p*}$.

where the strict positivity of $c^*(C^T(S), \beta, R, \kappa)(d, y) > 0$ follows from the Inada condition on $U_1(c; y^N)$ in its first argument, and we note the dependence of $c^*(C^T(S), \beta, R, \kappa)(d, y)$ on deep parameters for later reference. That proves the existence of a strictly positive greatest fixed point.

Step 4. *Fixed point comparative statics.* By standard fixed point statics argument for order continuous operators, the greatest fixed point $c^*(C^T(S), \beta, R, \kappa)(d, y)$ is increasing in $(C^T(S), \kappa)$, and decreasing in (β, R). Then, the associated stationary policy function for tradeable consumption conditioned on the equilibrium collateral constraint be fixed at $C^T(S) \in \mathbf{C}^f$ is given by

$$C(C^T(S), \beta, R, \kappa)(d, y) = \inf\{c^*(C^T(S), \beta, R, \kappa)(d, y), (1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(S))y^N\}$$

which is continuous in $(d, y) \in \mathbf{D} \times \mathbf{Y}$ by Berge's theorem, increasing in $(C^T(S), \kappa)$ and decreasing in (β, R). This completes to proof of the fixed point comparative statics claim in the lemma.

Step 5: $c^*(C^T(S), \beta, R, \kappa)(d, y)$ *the unique strictly positive fixed point.* This follows from an application of Corollary 4.1 in Li and Stachurski ([41]) for each $C^T \in \mathbf{C}^f$. To see this, for fixed $C^T \in \mathbf{C}^f$ put

$$\begin{aligned} \varsigma(d, y) &= u'((1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(d, y))y^N) \\ &= A_c(C^T)(d, y) \end{aligned} \tag{57}$$

and restrict the first step operator $A(c; C^T(S), \beta, R, \kappa)(d, y)$ to the set $\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma) \subset \mathbf{C}^p_{++} = \{c \in \mathbf{C}^p | c(d, y) > 0\}$, where

$$\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma) = \{c |\ c = \inf\{\hat{c}(d, y), \varsigma(d, y)), \hat{c} \in \mathbf{C}^p_{++}\}$$

where in our notation we make explicit the dependence of the the space $\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$ on the upper bound in (57). Let c_1 and c_2 be elements of $\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$. Equipped the space the $\mathbf{C}^p_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma(d, y))$ with the norm

$$\rho(c_1, c_2) = \parallel u'_\varsigma \circ c_1 - u' \circ c_2 \parallel$$

where $\parallel u' \circ c_1 - u' \circ c_2 \parallel < \infty$, where $u'(c) = U'(A(c))A_1(c^T, y^{NT})$ is strictly decreasing in c^T under Assumption 1, and give $\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e) \subset \mathbf{C}^p_{++}(\mathbf{D}^e \times \mathbf{Y}^e)$ its relative distance structure.

Clearly, from the arguments in Step 1 of this proof, $A(c; C^T(S), \beta, R, \kappa)(d, y)$ maps $\mathbf{C}^{p*}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$ into itself. By Li and Stachurski ([41], Proposition 4.1.a), the pair ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$,$\rho$) is a complete metric space. As $\beta R < 1$, by Li and Stachurski ([41], Proposition 4.1.c), for each $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is a contraction of modulus $0 < \beta R < 1$ in ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$,$\rho$). Then, by the contraction mapping theorem, $A(c; C^T(S), \beta, R, \kappa)(d, y)$ has exactly one fixed point in ($\mathbf{C}^{p*}_{++}(\mathbf{D}^e \times \mathbf{Y}^e; \varsigma)$,$\rho$). So, $c^*(C^T(S), \beta, R, \kappa)(d, y)$ is the unique strictly positive fixed point of $A(c; C^T(S), \beta, R, \kappa)(d, y)$ for $C^T \in \mathbf{C}^f$. ■

Corollary 12 *(Corollary of Lemma 6). The mapping $A^*(C^T; \beta, \kappa, R)(s^e))$ in (43) is order continuous on $\mathbf{C}^f$.*

Proof. As the operator in step 5 $A(c; C^T(S), \beta, R, \kappa)(d, y)$ is easily shown to be continuous in $C^T(S) \in \mathbf{C}^f$ in the topology of pointwise convergence, by the Bonsall-Nadler theorem on parameterized contractions, $c^*(C^T(S), \beta, R, \kappa)(d, y)$. (e. g., see Nadler ([56], Theorem 2 and Lemma, p. 581)). As $c^*(C^T(S), \beta, R, \kappa)(d, y)$ is also monotone increasing (in pointwise partial orders) by Step 4, $c^*(C^T(S), \beta, R, \kappa)(d, y)$ is order continuous on $\mathbf{C}^f$.

■

Proof of Theorem 7.

Proof. (i) Let $\Psi^*(R, \kappa, \beta) \subset \mathbf{C}^*$ be the set of fixed points of the mapping $A^*(C^T; \beta, R, \kappa)(s^e)) = \inf\{c^*(C^T(S); \beta, R, \kappa)(d, y),\ A_c(C^T)(s^e)\}$ defined in (43). That $A^*(C^T, \beta, R, \kappa)(s^e)) \in \mathbf{C}^*$ is immediate, as (a) by construction for fixed S, when $d = D$, $y = Y$, $c^*(C^T(S); \beta, R, \kappa)(d, y) \in \mathbf{C}^p$, and (b) when (d, y) is fixed, as $c^*(C^T(S); \beta, R, \kappa)(d, y)$ is increasing in $C^T \in \mathbf{C}^*$, and C^T is increasing in Y, and decreasing in D, $\inf\{c^*(C^T(S); \beta, R, \kappa)(d, y),\ A_c(C^T)(S))\}$ is increasing in Y, and decreasing in D. Further, if s^e is a collateral constrained state, then $d'_{A^*(C^T)}(s^e) = \kappa\{y^T + p(C^T(d,y))y^N)$ is increasing in y, and decreasing in d. By lemma 6. step 4, the operator $A^*(C^T)(s^e))$ is monotone increasing on $\mathbf{C}^*$. As $\mathbf{C}^*$ is a nonempty complete lattice, by Tarski's theorem (Tarski ([75], theorem 1), $\Psi^*(R,\kappa,\beta)$ is a nonempty complete lattice.

(ii) Noting its dependence on deep parameters now, as $A^*(C^T,\beta,R,\kappa)(s^e))$ is decreasing in (β,R), and increasing in κ, by Veinott's comparative statics version of Tarski's theorem (see Veinott ([77]), also see Topkis ([76], Theorem 2.5.2), the least and greatest selections of $\Psi^*(R,\kappa,\beta)$ exist as fixed points, and are decreasing in (β,R), and increasing in κ.

(iii) $A^*(C^T,\beta,R,\kappa)(s^e)) = \inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(s^e)\}$ is order continuous under pointwise partial orders on $\mathbf{C}^f$ as (a) $A_c(C^T)(s^e) = 1 + \frac{\kappa}{R})Y^T - D + \frac{\kappa}{R}p(C^T(d,y))y^N$ is pointwise continuous and monotone ($p(C)$ is continuous and monotone increasing under Assumption 1), (b) by Corollary 12 $c^*(C^T(S),\beta,R,\kappa)(d,y)$ is order continuous, and hence, (c) $\inf\{c^*(C^T(S);\beta,R,\kappa)(d,y),\ A_c(C^T)(s^e)\}$ pointwise continuous and increasing by Berge's theorem (as the state space is compact). Then the result follows from Dugundji and Granas ([28], p.15).

■

Proofs for Section 4

Proof of Theorem 11

We will prove the theorem using several preliminary lemmas. First, it will be shown that equation (55), when it holds with equality, generates a sequence of increasing level of debt d_+ for any d as long as $y^T = y_{lb}$ if y_{lb} is sufficiently small. Then, using this result, we show that starting from any initial condition, the collateral constraint will bind in finite time. This lemma will be useful to show the existence of an accessible atom in the third lemma. Then, the forth lemma show the existence of a unique invariant ergodic measure.

From now on we will assume that assumption 1 holds. Additionally, assume that in any SCE we have $d_{t+1} \leq H$ with $H \equiv \kappa(y_{ub} + P_{ub}y^N)$. Lemma 13 will show that, once the collateral constraint is imposed, H will not bind in equilibrium.

Lemma 13 *In equilibrium when $R\beta < 1$, we have $d_{++} > d_+ > d$ for any $d \in K_2$ if the collateral constraint does not bind, $y^T = y^T_+ = y_{lb}$ and $y_{lb} \in (0,\epsilon)$ with $\epsilon > 0$.*

Proof. Note that in equilibrium, WLOG it is possible to write $U'(A(c_1,c_2))A_1(x_1) \equiv u'(y^T - d + R^{-1}d_+)$. Then, under the assumptions stated in the lemma, it is clear that equation (55) can be written as:

$$u'(y^T - d + R^{-1}d_+) = R\beta \sum_{y^T_+} u'(y^T_+ - d_+ + R^{-1}d_{++})q(y^T_+)$$

Where $d_{++} \in K_2$. Suppose, to generate a contradiction, $d \in K_2$ and $d_+ \leq d$. Then, as $R > 1$ for ϵ sufficiently small, we have $u'(y^T - d + R^{-1}d_+) \longrightarrow u'_{ub}$, where u'_{ub} can be constructed using the definition of u' together with assumptions $1-a1$ and $1-c$ and $1-f$. Then, as $R\beta < 1$, $u'_{ub} > R\beta\sum_{y^T_+} u'(y^T_+ - d_+ + R^{-1}d_{++})p(y^T_+)$ which implies that d_+ is not optimal. Then, we must have $d_+ > d$ as desired. Replacing d with d_+, we get $d_{++} > d_+$. ■

Lemma 14 *For any $z \in Z$ the sequence $\{z, \phi_1, \phi_2, ...\}$, generated by (Z, P_φ), will hit the collateral constraint in finite time.*

Proof. Take any $y_0^T \in Y$, $d_0 \in K_2$ and a sequence with τ elements in $Y \times Y \times ... \times Y$ with $\{y_0, y_{lb}, ..., y_{lb}\}$. Then, the results in section 3 imply that, as long as the collateral constraint does not bind, $d_{\tau+1}(y_\tau^T, y_{lb}, ..., y_{lb}, y_0; d_0) = d'(y_\tau^T, d)$ and $P_\tau(y_\tau^T, y_{lb}, ..., y_{lb}, y_0; d_0) = P(d'(y_\tau^T, d))$, where the equalities follow from applying iteratively backwards the minimal state space policy function on equations (44) and (45) together with the envelope theorem, both derived in section 3. Note that the dependence of P on y^N has been omitted. Further, if the collateral constraint does not bind, we know from section 3 that $d'(y_\tau^T, d)$ / $P(d'(y_\tau^T, d))$ is decreasing /increasing in y_τ^T for each d. Further, $d'(y_\tau^T, d)$ / $P(d'(y_\tau^T, d))$ is increasing /decreasing in d for each y_τ^T. Lemma 16 below formally proves these claims. Then, using Lemma 13 we know that $\{d_1, ..., d_{\tau+1}\}$ / $\{P_0, ..., P_{\tau-1}\}$ is a (strictly) increasing / decreasing sequence which in turn implies that $g_t(y_t^T, y_{lb}, ..., y_{lb}, y_0; d_0) \equiv \kappa(y_t^T + P_t y^N) - d_{t+1}$ is a strictly decreasing sequence in t. To complete the proof we must show that: i) there exists a $y_\tau^T \in Y$ such that $g_\tau \leq 0$ and ii) $\tau < \infty$.
i) Suppose the collateral constraint does not bind. Then, $d_{t+1} \longrightarrow H$. By the definition of H and the fact that $d_{t+1} = d'(y_{lb}, d_t)$, we know that $|H - d_{t+1}| = H - d_{t+1} < \varepsilon$ for $t \geq N_\varepsilon$. For any given κ, we can take $\varepsilon \equiv \kappa(y_{ub} + P_{ub} y^N) - \kappa(y_{lb} + P_{ub} y^N) = \kappa(y_{ub} - y_{lb})$. Then, $d'(y_{lb}, d_t) > \kappa(y_{lb} + P_{ub})$, which is a contradiction. Then, the collateral constraint binds. That is, $g_\tau \leq 0$ for $g_\tau(y_{lb}, y_{lb}, ..., y_{lb}, y_0; d_0)$
ii) Simply take $\tau = N_\varepsilon$.
Note that H can be defined for any $y^T > y_{lb}$ which in turn implies that ε can be assumed to be arbitrarily small as desired. As the initial conditions were arbitrary, the proof is completed. ■

Before stating and proving the next lemma we need some additional notation. Let $z \in Z$. Then, any solution to the system defined by equations (46) to (50) will be denoted $z(d, y^T) \equiv z = [d \quad y^T \quad y^N \quad c_1 \quad c_2 \quad p \quad m]$.

Lemma 15 *Let $J_1 \equiv Z$, where Z was defined in section 4.1. There is a point $d_* \in K_2$ with $d_{ub} > d_* > d_{lb}$ and a selection $\varphi \in \Phi$, where Φ is the equilibrium correspondence which contains all Generalized Markov Equilibria, such that for any $(y_0^T, d_0) \in Y \times K_2$, there is a sequence $\{\phi_0, \phi_1, \phi_2, ...\}$, generated by (J_1, P_φ), which satisfies $\phi_\tau = z(d_*, y_{lb}) \in J_1$ with $\tau < \infty$.*

First, some notation and a auxiliary lemma. Let $d'(d_0, y_{lb})$ be the policy function obtained from solving the optimization problem in the RE defined in section 3 for the unconstrained case. It can be seen from the results in section 3 that, as long as we are dealing with the unconstrained problem, $d'(d_0, y_{lb})$ is independent of prices and, thus, we can take this policy function for any c^T. Then, $\varphi \in \Phi$ satisfies:

$$d'(d_*, y_{lb}) = \kappa \left[y_{lb} + y^N \left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb})/R) - d_*)} \right) \right] \tag{58}$$

$$U' \left\{ A_1 \left(y_{lb} + (d'(d_*, y_{lb})/R) - d_* \right) \right\} = \beta R E_\varphi[-d'(d_*, y_{lb})] \tag{59}$$

Where φ is defined by taking any vector $d''(y') \in K_2$ [58] for any $y' \in Y$ such that:

$$U' \left\{ A_1 \left(y_{lb} + (d'(d_*, y_{lb})/R) - d_* \right) \right\} = \beta R \textstyle\sum_{y'} [U' \left\{ A_1 \left(y' + (d''(y')/R) - d'(d_*, y_{lb}) \right) \right\}] q(y')$$

Before proving Lemma 15, we need a preliminary lemma. It simply proves that, in an unconstrained framework, in partial equilibrium, more "disposable income" means more consumption and less debt.

[58] We state d'' solely as a function of y' to keep the notation simple. It is possible to construct a selection $d''(d, y, d', y')$ in order to ensure the optimality of the atom for any possible path. We compute it in the numerical section (see the supplementary appendix for section 5.3).

Lemma 16 *Let $y^D \equiv y^T - d$ and $c_1\left(y^D\right)$ be a tradable optimal unconstrained consumption. Then, in any unconstrained equilibrium, $y^D > \widetilde{y}^D$ implies i) $c_1\left(y^D\right) > c_1\left(\widetilde{y}^D\right)$ and ii) $d'\left(y^D\right) < d'\left(\widetilde{y}^D\right)$*

Proof. i) Suppose not. Then, $y^D > \widetilde{y}^D$ and $c_1\left(y^D\right) \leq c_1\left(\widetilde{y}^D\right)$. Then, the budget constraint in any unconstrained equilibrium implies $d'\left(y^D\right) < d'\left(\widetilde{y}^D\right)$. As $c_1\left(y^D\right)$ and $d'\left(\widetilde{y}^D\right)$ are optimal, we have:

$$U'\left(A_1(y^D)\right) \geq U'\left(A_1(\widetilde{y}^D)\right) = \beta RE\left(-d'\left(\widetilde{y}^D\right)\right) > \beta RE\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal.

ii) Let $y^D > \widetilde{y}^D$ and $c_1\left(y^D\right) > c_1\left(\widetilde{y}^D\right)$. Assume, in way of contradiction $d'\left(y^D\right) \geq d'\left(\widetilde{y}^D\right)$. Then:

$$U'\left(A_1(y^D)\right) < U'\left(A_1(\widetilde{y}^D)\right) = \beta RE\left(-d'\left(\widetilde{y}^D\right)\right) \leq \beta RE\left(-d'\left(y^D\right)\right)$$

Which implies a contradiction as $c_1\left(y^D\right)$ and $d'\left(y^D\right)$ are assumed to be optimal. ■

Lemma 17 *In any unconstrained equilibrium, there is a decreasing sequence of debt with increasing tradable consumption for the same level of tradable output, y^T*

Proof. Let $d'(d_t, y_t^T) = d_{t+1}$. Thus, $U'\left(A_1(y_t^T - d_t + (d_{t+1}/R))\right) = \beta RE\left(-d_{t+1}\right)$. Take $\widetilde{d}_t < d_t$. Then $U'\left(A_1(y_t^T - \widetilde{d}_t + (d_{t+1}/R))\right) < \beta RE\left(-d_{t+1}\right)$. Then, there exist $\widetilde{d}_{t+1} < d_{t+1}$ with $(\widetilde{d}_{t+1}/R) - \widetilde{d}_t > (d_{t+1}/R) - d_t > 0$ such that $U'\left(A_1(y_t^T - \widetilde{d}_t + (\widetilde{d}_{t+1}/R))\right) = \beta RE\left(-\widetilde{d}_{t+1}\right)$. By letting $d_{t+1} = \widetilde{d}_t$, $\widetilde{d}_{t+1} = d^*_{t+2}$ and $\widetilde{d}_t = d^*_{t+1}$, $d_t = d^*_t$, we obtain the decreasing sequence $\left\{d^*_{t+i}\right\}_i$ which generates a increasing consumption sequence $\left\{c^*_{1,t+i}\right\}_i$ because $(\widetilde{d}_{t+1}/R) - \widetilde{d}_t > (d_{t+1}/R) - d_t > 0$ and $y^T_{t+i} = y^T$ for all i. ■

Lemma 18 *The selection $\varphi \in \Phi$ exists.*

Proof. Lemma 16 implies that there exist $d_* \in K_2$ such that:

$$d'(d_*, y_{lb}) = \kappa\left[y_{lb} + y^N\left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(d_*, y_{lb})/R) - d_*)}\right)\right]$$

This is possible as the LHS / RHS of this equation is increasing / decreasing in d, both sides are continuous functions of d according to the results in section 3.2 and

$$d'(0, y_{lb}) < \kappa\left[y_{lb} + y^N\left(\frac{A_2(y^N)}{A_1(y_{lb} + (d'(0, y_{lb})/R))}\right)\right]$$

That is the collateral constraint does not bind if debt is non-positive.

Take any $y_0^T, d_0 \in J$ and a sequence $\{y_0^T, y_{lb}, ..., y_{lb}\}$ of $\tau + 1$ elements. The results in Lemma 13 imply that, as long as the collateral constraint does not bind in period $\tau < \infty$ (this assumption can be imposed WLOG due to Lemma 14), there is a constant sequence of tradable consumption $\{c_{1,0}, ..., c_{1,\tau-1}\}$ with $c_1(y_0^T, d_0) = c_{1,t}$ for $0 \leq t \leq \tau - 1$ which can be implemented as a SCE. Further, as it is shown in Lemma 16, it is possible to choose $c_1(y_0^T, d_0)$ to be decreasing in d_0 and increasing in y_0^T. Equipped with these paths we will deal with a fraction of all possible initial conditions in $Y \times K_2$. In order to deal with the rest of the space we will use 13 and the inequality nature of the Euler equation to deal with the constrained case, if necessary.

Now take $y_0^T = y_{lb}$ and $d_0 = d_*$ with $\kappa(y_{lb} + P(x_1(y_{lb}, d_*))y^N) = d'(y_{lb}, d_*)$. The existence of d_* follows from Lemmas 15 and 18. This point defines the "atom" and we must show that there exist a positive probability sequence starting from any initial condition that hits it in finite time. Intuitively, the atom will be defined as the level of wealth, d, for which the collateral constraint binds with equality at $t = 0$ for the lowest possible level of current income, y_0^T. Thus, the strategy of the proof is to show that regardless of the initial condition, it is possible to construct a positive probability path that will hit the constraint: a) later, b1) with a bigger tradable consumption level and with more debt, both today or b2) with more debt tomorrow (and not necessarily with more consumption today). Call this last point $z(d_\tau, y_\tau)$. First, note that, as the collateral constraint hits with equality in the atom, for any other path, we will have more debt but not necessarily more consumption. In any case, the euler inequality implies that $z(d_*, y_{lb})$ satisfies the system of equations which defines the GME due to the strict inequality in the primal formulation of this equation. That is, the qualitative properties of $z(d_\tau, y_\tau)$ imply, due to the inequality in the Euler equation in the primal characterization of the sequential equilibria coupled with the backwards nature of the definition of any GME, that $z(d_\tau, y_\tau)$ and $z(d_*, y_{lb})$ are both a solution to the constrained system of equations which define the GME. This last claim is proved in Lemma 18.
We will now show that the chain will hit the atom, $z(d_*, y_{lb})$, starting from any initial condition in $J = Y \times K_2$. We will proceed in 2 regions: i) $d_0 < d_*$ and $y_0^T = y_{lb}$, ii) $d_0 > d_*$ and $y_0^T > y_{lb}$. Intuitively, region i) insures, due to Lemma 16, that the collateral constraint will not bind at $t = 0$ and that initial tradable consumption is bigger than the atom level. Using Lemma 13 and 14 we will construct a positive probability sequence with increasing debt which insures the "reversion" to the atom at the time the path hit the constraint. Region ii) has 2 possible sub-regions. ii.a) $y_0^T - d_0 > y_{lb}^T - d_*$, in which case again due to Lemma 16 we will have bigger initial consumption and smaller debt when compared with the atom level. Thus, we can construct a sequence with increasing debt in an unconstrained environment using the arguments in region i) until we exceed the atom's level and revert to it when the path hit the collateral constraint. ii.b) $y_0^T - d_0 < y_{lb}^T - d_*$, in which case the consumption level is smaller when compared to the atom level due to Lemma 16. Thus, we must construct an increasing consumption sequence in order to revert to the atom. In this case there are multiple possibilities: the constraint can be bigger (ii.b.1) or smaller (ii.b.2) compared with the atom level. In region ii.b.1 an increase in consumption will take place in a constrained environment if p is not sufficiently sensitive or elastic (ii.b.1.1). Contrarily, if prices are sufficiently elastic, an increase in consumption will take place in an unconstrained environment (ii.b.1.2). Finally, in region ii.b.2, when the constraint is bigger with respect to the atom level, as debt is also bigger we may have a constrained or an unconstrained regime. In these cases we can use the results in regions ii.b.1 to generate an increasing consumption level until we reach the atom consumption.
i) For $d_0 < d_*$ and $y_0^T = y_{lb}$, we have $\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N)$ and $d'(y_{lb}, d_0) < d'(y_{lb}, d_*)$ from Lemma 16 which, using Lemma 14, implies that for the sequence $\{\phi_0, ..., \phi_t\}$ the chain will hit the collateral constraint in $t = \tau > 0$ with $d_\tau > d_*$.
We claim that the system of equations given by (46) to (50) can also be solved by $z(d_*, y_{lb})$ and thus we have that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory. To prove this claim, note that from the definition of an equilibrium correspondence, we have that any d in z with $z \in Z$ that solves $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_0))y^N) - d)\} \geq E(m_+)$ is a predecessor of $z_+(y_+^T)$ for any $y_+^T \in Y$. As $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} > E(m_+)$, equations (46) to (50) imply that $\{\phi_0, ..., z(d_*, y_{lb})\}$ is an equilibrium trajectory as desired.
ii) For $d_0 > d_*$ and $y_0^T > y_{lb}$.
Region ii.a): $y_0^T - d_0 > y_{lb} - d_*$. As y^D is bigger in this case when compared with the atom level and we have more tradable output, $d'(y_0^T, d_0) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < \kappa(y_0^T + P(c_1(y_{lb}, d_0))y^N)$ and we can use the arguments in region i).
Region ii.b): $y_0^T - d_0 < y_{lb} - d_*$.
Region ii.b.1): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. That is, the collateral hits at $t = 0$ with $c_1(y_{lb}, d_0) > c_1(y_0^T, d_0)$ which implies that we must generate an increasing sequence of consumption from the constrained regime. Note than when we increase tradable consumption, depending on the sensibility of p with respect to c^T, we may enter into a constrained (ii.b.1.1) or an unconstrained regime (ii.b.1.2).
Region ii.b.1.1): $1 \geq \kappa y^N R^{-1} p'$, where p' is the derivative of (46) with respect to d_+. As $c_1(y_{lb}, d_*) >$

$c_1(y_0^T, d_0)$ and

$$\begin{aligned}\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) &< \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) \\ &< d'(y_0^T, d_0)\end{aligned}$$

we have:

$$\begin{aligned}U'\{A_1(y_0^T + R^{-1}\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) - d_0)\} &> U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} \\ &= E(m_+; -\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)) \\ &> E(m_+; -\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N))\end{aligned}$$

The above inequality implies that any path in this region with $P(c_1(y_t^T, d_t)) < P(c_1(y_{lb}, d_*))$ is optimal. By setting $\{y_0^T, y_{lb}, ..., y_{lb}\}$ we can construct an increasing sequence:

$$\{P(c_1(y_0^T, d_0)), P(c_1(y_{lb}, d_1)), ..., P(c_1(y_{lb}, d_\tau))\}$$

converging to $P(c_1(y_{lb}, d_*))$ as desired.
Region ii.b.1.2): $1 < \kappa y^N R^{-1} p'$. In this case an increase in consumption take us to the unconstrained region. Using Lemma 17, we can generate a path $\{y_0^T, y_1^T, ..., y_\tau^T\}$, with $y_t^T > y_{lb}$; $t = 1, ..., \tau$, of increasing consumption and decreasing debt until we get $c_1(y_{lb}, d_*) < c_1(y_\tau^T, d_\tau)$. Then, set $\{y_\tau^T, y_{lb}, ..., y_{lb}\}$ The argument in region i) insure that there is a finite time, $\tau + \tau_1$, such that the path hits the atom.
Region ii.b.2): $\kappa(y_0^T + P(c_1(y_0^T, d_0))y^N) > \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) < d'(y_0^T, d_0)$. In this case, we can be either in the unconstrained or constrained case. For the former, we can use the same arguments as in section ii.b.1.2) as we need a path of decreasing debt and increasing consumption. For the latter, however, note that by setting $\{y_0^T, y_{lb}, ..., y_{lb}\}$ we enter into the ii.b.1) region for $t > 0$ as $\kappa(y_{lb} + P(c_1(y_{lb}, d_t))y^N) < \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)$ as $d_t > d_*$. ■

Lemma 19 *the results in Lemmas 13 to 15 imply that there exists a selection $\varphi \sim \Phi$ and a Markov process (J_1, P_φ) that has an accessible atom, $z(d_*, y_{lb})$, and is $P_\varphi(z(d_*, y_{lb}), .)-$irreducible*

Proof. Follows directly from proposition 1. ■

Lemma 20 *Let (J_1, P_φ) be the process defined in Lemma 15. If the collateral constraint hits at time $\tau > 0$ with ($d_\tau > d_*$ and $c_{1,\tau} > c_1(y_{lb}, d_*)$) or with ($d_{\tau+1} > d(d_*, y_{lb})$ and $d_\tau \leq d_*$), then $\{\phi_0, \phi_1, ..., \phi_{\tau-1}, z(d_*, y_{lb})\}$ is an equilibrium trajectory.*

Proof. If the collateral constraint binds for consumption $c_{1,\tau}$ and debt d_τ, then it must satisfy $U'\{A_1(y_\tau^T + R^{-1}\kappa(y_\tau^T + P(c_{1,\tau})y^N) - d_\tau)\} \geq E(m_+)$. The conditions in the remark imply that $U'\{A_1(y_{lb} + R^{-1}\kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N) - d_*)\} > E(m_+)$ as desired. Next note that $d_{\tau+1} > d(d_*, y_{lb})$ implies $\kappa(y_\tau + P(c_1(y_\tau, d_\tau))y^N) > \kappa(y_{lb} + P(c_1(y_{lb}, d_*))y^N)$. Assume $c_1(y_\tau, d_\tau) \leq c_1(y_{lb}, d_*)$. Then:

$$y_\tau + R^{-1}d_{\tau+1} - d_\tau \leq y_{lb} + R^{-1}d(d_*, y_{lb}) - d_*$$

But this implies a contradiction since $d_{\tau+1} > d(d_*, y_{lb})$, $y_\tau \geq y_{lb}$ and $d_\tau \leq d_*$. Then, $c_1(y_\tau, d_\tau) > c_1(y_{lb}, d_*)$ and we are back in the previous case.
■

Lemma 21 *Let (J_1, P_φ) be the Markov process in Lemma 15. Then, (J_1, P_φ) has a unique, ergodic, invariant probability measure.*

Proof. Note that Lemma 15 imply that $P^{\tau}_{\varphi}(z(d_*, y_{lb}), \{z(d_*, y_{lb})\}) > 0$ with $\tau < \infty$. Given the results in Remark 4.2.1, proposition 4.2.2, theorem 8.2.1 and theorem 10.2.1 in Meyn and Tweedie ([52]) imply that (J_1, P_{φ}) has an unique invariant measure. As $\tau < \infty$ for any initial condition in J_1, theorem 10.2.2 in Meyn and Tweedie ([52]) implies that the invariant measure is a probability measure. As it is unique, the Krein-Milman theorem (See Futia, [30]) implies that this measure is ergodic. ■

Supplementary Material for Section 5.1

Up to now, in this sub-section, we have been silent about optimality. Assume that for some exchange rate level, P_0, there is a debt level d_0 which satisfies equation (50) for $y^T = y_{lb}$. That is, d_0 satisfies: $U'\{A_1(y_{lb} + R^{-1}\kappa\{y_{lb} + P_0 y^N\} - d_0)\} \geq\ E(m_+)$. Taking into account that we are assuming that (d, p) belong to a compact set, the existence of d_0 and P_0 follows without loss of generality given theorem 3 in section 2. Note that any $d = d_0 + (y^T - y_{lb})(1 + \kappa/R)$ will also satisfy equation (50). Now, the definition of $d_2(y^T)$ implies: $d_2(Y) - d_2(Ymin) = (Y - Ymin)(1 + \kappa/R)$. As can be seen in figure 3, if any d_0 is in the locus formed by $f(p) = K(d)$ (i.e. which satisfies equation (51)) with $(d, P) >> 0$, above the zero-consumption line $d_2(Ymin) - P(Ymin)$ line (i.e. which is above equation (52)) and satisfies equation (50), then $d = d_0 + (Y - Ymin)(1 + \kappa/R)$ is in the locus formed by (51) and above (52) but with the last equation crossing the d-axis at $d_2(Y)$. By replacing Y with $y^T > y_{lb}$ and $Ymin$ with y_{lb}, we know that d will also be feasible and optimal if d_0 is so: it will be to the right of the vertical axis, above the $d_2(y^T)$-$P(y^T)$ line, over the locus formed by $f(p) = K(d)$ (i.e. it will be feasible) and will satisfy equation (50) (i.e. it will be optimal). Finally, we must show that any the pair (d, P) satisfies the optimality requirement (i.e. it satisfies (50)) and find a contour in the plane in order to separate optimal from non-optimal pairs. We will start with (d_0, P_0). Take any $m_+(y^T_+)$ from Z which satisfies equation (49) with $d_+ = \kappa\{y_{lb} + P_0 y^N\}$ and any given d_{++}. Equation (60) below, when it holds with equality, defines a locus increasing in the (d, P) plane which determines optimality. Call this map $h(d, P) = 0$. Take any (d_0, P_0) over (60). Any $P \leq P_0$ is optimal because of the weak inequality in the primal optimization problem in the SCE. Thus, optimality lies below (60). Moreover, note that this locus in increasing in d_{++}. Thus, $h(d, P; d_{++}) = 0$ As can be seen from the definition of GME, $d_{++}(y_+)$ for each $y_+ \in Y$ defines a selection of the equilibrium correspondence and, thus, 1 of possible multiple GME. As the upper bound on d_{++} can be chosen freely [59]. Then, there is always a value in $[d_l, d_u]$, the set which contains d by assumption, which insures optimality. That is, which satisfies:

$$U'A_1\left(y^T + R^{-1}\kappa\left(y^T + Py^N\right) - d\right) \geq \beta RE[m_+\left(-R^{-1}\kappa\left(y^T + Py^N; d_{++}\right)\right)] \tag{60}$$

Figure 4 illustrates these facts. Note that the multiple displacements of (51) in solid lines form the equilibrium correspondence of some GME for an arbitrarily large selection $\varphi \in \Phi$. That is, figure 4 below presents the full picture as it combines the feasible pairs in figure 3 with the optimal ones in $h(d, P; d_{++}) = 0$. Note that there is a fraction of equation (51) which intersects with $d_2(Ymin) - P(Ymin)$ that is depicted in light grey. This is because it evolves $d < 0$. As the level of P becomes smaller, c^T does so and it is not necessary to take debt to satisfy optimality. In other words, the light grey areas represent the unconstrained regime for $y^T = y_{lb}$. A similar argument holds for $y^T = Ymax \equiv y_{ub}$. As it is shown in the appendix for section 4 (see Lemma 16), in the unconstrained regime, tradable consumption is increasing in tradable output, while debt is decreasing in the same variable. Thus, it is possible that $d(d, Ymax) < p(Ymax - d + d(d, Ymax)/R, y^N)$, where $p(.,.)$ is equation (44) and $d(.,.)$ is the policy function for the unconstrained regime associated with the envelope m_+. The locus formed by equation (51) which intersects with $d_2(Ymax) - P(Ymax)$ is depicted in light grey as the pair $(d + (Ymax - Ymin)(1 + \kappa/R), PYmax)$, where $f(PYmax) = K(d + (Ymax - Ymin)(1 + \kappa/R))$ may belong to the unconstrained regime given the fact that $Ymax$ can be chosen to be arbitrarily large. Further,

[59] For expositional purposes, in this subsection we are assuming that endogenous variables lie in a compact set. Thus, the existence of an arbitrarily large upper bound can be insured without loss of generality. However, in section 2, we showed that it depends on the bounds on marginal utility (see Table 1).

note that there is a point for which $d(d, Y^T) = p(y^T - d + d(d, y^T)/R, y^N)$, with $Ymin < y^T \leq Ymax$ (see Lemma 14 in the appendix). Thus, as $p(.,.)$ is continuous and $d(.,.)$ are continuous and decreasing in y^T, and $c^T = Y^T - d + d(d, Y^T)/R$ is increasing in Y^T, we know that the collateral constraint will be binding for any $y < Y^T$. Thus, the desired result follows.

Supplemantary material for section 5.2 (Fisherian deflation)

In order to show the existence of Fisherian deflation, we must prove that there exist a *binding* level of debt d_{++} which simultaneously satisfies equations (46) and (53) with $0 < d_{++} < \kappa\{y_{lb} + p_* y^N\} = d_{*,+}, c_+ < c_*$ and $p_+ < p_*$. That is, $d_{++} = \kappa\{y_+^T + p_+ y^N\}$ must satisfy:

$$p = \frac{A_2(y^N)}{A_1(y_+^T + R^{-1}\kappa\{y_+ + p_+ y^N\} - \kappa\{y_{lb} + p_* y^N\})} \tag{61}$$

By setting $y_+^T > y_{lb}$, it is possible to choose $\kappa\{y_+^T + p_+ y^N\} < \kappa\{y_{lb} + p_* y^N\}$ and c_+ will remain positive. Note that the last 2 inequalities together imply $p_+ < p_*$. Moreover, p is increasing and real valued in c, which imply that we must have $c_+ < c_*$. Finally, to show optimality, d_{++} must satisfy:

$$\begin{aligned} U'\{A_1(y_+^T + R^{-1}\kappa\{y_+ + p_+ y^N\} - \kappa\{y_{lb} + p_* y^N\})\} \geq \\ \geq \ E(U'\{A_1(y_{++}^T - \kappa\{y_+ + p_+ y^N\} + R^{-1}\overline{d})\}) \end{aligned}$$

As $c_+ < c_*$, we know $U'\{A_1(c_+)\} > U'\{A_1(c_*)\}$. As $y_+ + p_+ y^N < y_{lb} + p_* y^N$, it follows that $E(U'\{A_1(y_{++}^T - \kappa\{y_+ + p_+ y^N\} + R^{-1}\overline{d})\}) < E(U'\{A_1(y_{++}^T - \kappa\{y_{lb} + p_* y^N\} + R^{-1}\overline{d})\})$, which implies that the condition above is satisfied. As mentioned before, the path of Spain in the 2008 sudden stop was characterized by a monotonically decreasing consumption sequence; even after the sharp drop in period "T". At the same time, we observed an improvement in the current account. The "Fisherian deflation" described above matches these facts as $d_{++} < d_+$, which implies the observed improvement in the current account, and $c_+ < c_*$, which gives us the decreasing consumption sequence.

Supplementary Material for section 5.3 (numerical procedure)

We now present the ergodic, stationary and non-stationary algorithm together with some details pf the procedure involved.

GME Ergodic Algorithm

Step 1: Computation

- Fix the vector of parameters from section 2.1, $[\kappa, \beta, \sigma, \xi, a] \equiv \Theta \gg \overrightarrow{0}$ with $U(c_1, c_2) = \frac{A(c_1,c_2)^{(1-\sigma)}}{1-\sigma}$ and $A(c_1, c_2) = (a(c_1)^{(1-1/\xi)} + (1-a)(c_2)^{(1-1/\xi)})^{\frac{1}{1-1/\xi}}$.
- Fix $Y \times K_1 \times K_2$
- Compute $d'(d, y)$ from (18) *ignoring* the collateral constraint.
- Compute d_* from equation (58) in the appendix
- Compute $\varphi \in \Phi$ from equation (59) in the appendix

Step 2.1: Stationary simulation

- Take a "draw" of length $T+1$ from (Y,q), the exogenous Markov process which generates tradable output.
- Fix $[d_{T-1}, y_{T-1}]$ from $Y \times K_2$, obtain d_T from $d'(d,y)$. Verify if the collateral constraint binds and adjust d_T if necessary. Then compute $d_{T+1}(y_T, d_{T-1}, y_{T-1})$ from $\varphi \in \Phi$ for every $y_T \in Y$. Compute the rest of the endogenous variables from equations (46)-(50).
- Take $[p_T(y_T), d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}), d_{T-1}(y_{T-2})]$ from equations (46)-(50). Note that d_{T-1} is in the preimage of $d'(d,y)$ with $d_T = d'(d_{T-1}, y_{T-1})$.
- Repeat the above • until you get $[p_0(y_0), d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.

Step 2.2: Ergodic simulation

- Take a "draw" of length $N+1$ from (Y,q), the exogenous markov process which generates tradable output.
- Compute $[p_j(y_j), d_j(y_{j-1})]$ for $j = N+1, N, N-1$ as in the stationary simulation procedure
  - If d_N binds, $y_{N-1} - d_{N-1} + d_N/R > y_{LB} - d_* + d'(d_*, y_{LB})/R$ and $d_N > d'(d_*, y_{LB})$, the process hits the atom and reverts to it.
- Continue until $[p_0(y_0), d_0]$

The numerical procedure is based on the policy function of the RE, $d'(.,.)$. It is computed as a solution to (18) without a collateral constraint. Then the atom d_* is computed using $d'(.,.)$ in (58). Note that if the process hits the constraint in period t, then $d'(d_t, y_t) > d'(d_*, y_{lb})$. Thus in order to find a regeneration point, we need:

$$U'\left\{c_1\left(y_t - d_t + d'(d_t, y_t)/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_t, y_t) + d''(d_t, y_t, d', y')\right)\right\} \equiv f(d''(y'); d, d', y_t) \geq 0$$

$$U'\left\{c_1\left(y_{lb} - d_* + d'(d_*, y_{lb})/R\right)\right\} - \beta R \sum_{y'} U'\left\{c_1\left(y' - d'(d_*, y_{lb}) + d''(d_*, y_{lb}, d', y')\right)\right\} \geq 0$$

Note that in section 4 we require d'' to be independent of d, d', y. This fact implies that we can easily find d'' but there are some additional conditions which must be satisfied by paths in order to revert to the atom (i.e. consumption and debt must be greater than the atomic level). This restrictions affects the frequency at which the atom is hit. Thus, in order to improve the recurrent structure of sets and gain computationally efficiency, we modified the selection from the GME. When we allow d'' to depend on an expanded state space, we can find a stationary Euler equation for each d, y as d' has at most 2 solutions when the collateral constraint is binding.

To formally prove the existence of this selection is beyond the scope of this paper. We need to show that $f(d''(y'); d, d', y_t) \geq 0$ has a minimum in d'' for each $y'; d, d', y_t$, subject to the collateral constraint, and that the value function of this problem is equal to zero if the collateral constraint is not binding. Numerically, this selection is easily implemented and the algorithm is really fast as we do not need to compute every selection of the GME.

GME Non-Stationary Algorithm

Step 1

- Fix the vector of parameters from section 2.1, $[\kappa, \beta, \sigma, \xi, a] \equiv \Theta \gg \overrightarrow{0}$
- Fix $Y \times K_1 \times K_2$ in order to define a compact set for (y, p, d) respectively.
- Fix a $[p_{T+2}(y_{T+2}), d_{T+2}(y_{T+1})]$ for each $(y_{T+1}, y_{T+2}) \in Y \times Y$. This will define a selection $\varphi \in \Phi$ using the Euler equation in time T.

Step 2

- Take a "draw" of length $T + 1$ from (Y, q), the exogenous markov process which generates tradable output.
- Fix $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ from $K_1 \times K_2$ and compute $[p_T(y_T), d_T(y_{T-1})]$ from equations (46)-(50)
- Take $[p_T(y_T), d_T(y_{T-1})]$ as given from the previous • and compute $[p_{T-1}(y_{T-1}), d_{T-1}(y_{T-2})]$ from equations (46)-(50). Include $[p_{T+1}(y_{T+1}), d_{T+1}(y_T)]$ in the Euler equation. This step implies a change in the selection from Φ and, thus, breaks the stationarity of the process.
- Repeat the above • until you get $[p_0(y_0), d_0]$, where d_0 is allowed to be independent of y_0 as they are both initial conditions of the system.

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